# Analytic Hierarchy Process using Belief Function Theory: Belief AHP Approach 

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## Introduction

Given the complexity of our life today, people have to make lots of decisions during their everyday life. Some decisions may be made considering a single criterion, but these are very limited to the simple and relatively unimportant ones. Therefore, the two terms "multi-criteria" and "decisionmaking" are nearly inseparable, especially when making complex decisions that require consideration of all the different aspects.

The problems considered in this master thesis consist of a finite set of alternatives which are evaluated on the basis of multiple, usually conflicting, criteria are called Multi-Criteria Decision Making (MCDM) problems.

Within the framework of MCDM, there exists a large amount of methods each designed to tackle certain specificities of real-life MCDM problems. We have then two large families of methods. On the one hand, the outranking approach introduced by Roy where some methods like Electre and Promothee are developed (Brans et al., 1986) (Figueira et al., 2005). On the other hand, the value and utility theory approaches mainly started by Keeney and Raiffa (Keeney \& Raiffa, 1976), and then implemented in a number of methods (Triantaphyllou, 2000).

In the recent years, it has become more than apparent that MCDM methods should be able to take into account uncertainty and imprecision in the parameters. Hence, the classical methods applying both multi-attribute utility theory and outranking model do not accomplish this. In order to overcome this limitation, many researches have been done to adapt standard approach to this kind of environment. The idea was to introduce theories managing uncertainty and/or imprecision, such as probability theory, evidence theory and fuzzy set theory in the development of these MCDM methods.

Our aim, through this work, is to investigate the Analytic Hierarchy Process (AHP) which is one of the most well-established and frequently used
methods for solving a MCDM problem, due to its good results and to its simplicity. It provides a structured simple solution to decision making problems. Thus, this approach has gained wide popularity in different applications such as country risk evaluation, portfolio selection and management, water resources planning, highway planning, job evaluation, etc.

However, the AHP method uses precise estimates of the decision maker. This condition cannot be satisfied in many applications because judgments elicited from experts are usually imprecise and unreliable due to the limited precision of human assessments.

Therefore, we propose to develop a new method for solving a multi-criteria decision problem in the framework of AHP under condition that the decision maker may express his preferences with some degrees of uncertainty.

At first, we propose a method that deals with uncertainty in two levels: the criterion and alternative levels. On the one hand, our proposed approach allows the decision maker to express the importance of criteria with incomplete and imprecise preferences. So, the decision maker determines his opinions on groups of criteria instead of single ones. On the other hand, our method is able to use sets of criteria to compare sets of alternatives, which can help the decision maker to express subjective judgments between these alternatives. Then, we are interested in treating the uncertainty that may appear in the comparison procedure. Hence, to evaluate the responses of the pair-wise comparison question, the decision maker expresses his judgment with some degrees of uncertainty. So, this approach deals with belief pair-wise comparison matrix.

In both methods, the uncertainty will be managed using the belief function theory as interpreted in the Transferable Belief Model (TBM). The choice of the TBM seems appropriate as it provides a convenient framework for dealing with incomplete and uncertain information, notably those given by experts. This theory is chosen because it has a powerful evidence combination rule, and it represents properly partial and total ignorance; it assigns beliefs to individual elements of the hypothesis set as well as their subsets.

Finally, to illustrate the feasibility of our approaches and to judge their performances, we have applied our proposed methods on a real application problem: The life cycle assessment. In fact, we have considered the PVC (Polyvinyl chloride) life cycle especially the end of life phase. The challenge facing an expert here is the choice of the country where the environmental
impact is the least important for the destruction of a kilogram of PVC.
This report is organized in four chapters belonging to two main parts:
Part I: Theoretical aspects. This part presents the necessary theoretical aspects regarding the belief function theory and the AHP method which are detailed respectively in chapter 1 and chapter 2 .

Part II: Belief AHP method details our proposed methods namely Belief AHP methods. Chapter 3, details the different steps that we have developed relatively to the building procedure of the MCDM within an uncertain context. Chapter 4 deals with implementation and application of our proposed approaches on a real application problem.

Finally, a conclusion summarizes all the work presented in this report and proposes further works that may be done to improve our method.

## Part I

## Theoretical Aspects

## Chapter 1

## Belief function theory

### 1.1 Introduction

The belief function theory, sometimes called evidence theory or DempsterShafer theory, is considered as a useful theory for representing and managing uncertain knowledge. This theory is introduced as a model to represent quantified beliefs (Shafer, 1976).

The term Dempster-Shafer refers to the origin of the theory. In fact, in the 1960s, Dempster developed the basic ideas of a new mathematical theory of uncertainty that includes a kind of upper and lower probabilities (Dempster, 1967) (Dempster, 1968). Then, in 1970s, it was extended by (Shafer, 1976) to what is now known as belief function theory.

Several interpretations of this theory have been proposed (Smets, 1991) among them: the Dempster's model (Dempster, 1967) (Dempster, 1968), the lower probability model (Roy, 1996), the theory of hints (Kohlas \& Monney, 1995) and the Transferable Belief Model (TBM) (Smets \& Kennes, 1994).

In this master thesis, we deal with the interpretation of the belief function theory as explained by the TBM. In this model, beliefs can be held at two levels: a credal level where beliefs are quantified and entertained, and a pignistic level, where beliefs are used for decision making. In each level, the TBM provides several tools for representing and managing beliefs under uncertainty (Smets \& Kennes, 1994).

In this chapter, the basic concepts of this theory are introduced. After
that, we will describe some special belief functions. Then, we will present some notations like the combination, the discounting, etc.

### 1.2 Basic concepts

In this Section, we are going to present the main concepts underlying the belief function theory. There are three important functions: the basic belief assignment function (bba), the belief function (bel), and the plausibility function ( $p l$ ).

### 1.2.1 Frame of discernment

In belief function theory, $\Theta$ is a non empty set which contains all the possible elements of interest in each particular context and its elements are exhaustive and mutually exclusive events (Smets \& Kennes, 1994). $\Theta$ is called the frame of discernment or the universe of discourse. It is also the initial set of possible states in the problem domain.

All the subsets of $\Theta$ belong to the power set of $\Theta$, denoted by $2^{\Theta}$, and every element of $2^{\Theta}$ is called a proposition or an event.

$$
\begin{equation*}
2^{\Theta}=\{A / A \subseteq \Theta\} \tag{1.1}
\end{equation*}
$$

In the Shafer's model, $\Theta$ is assumed to be exhaustive (Shafer, 1976) which means that the solution to a given problem is unique and is necessarily included in this frame of discernment. However, in the TBM, Smets relaxed this condition, considering that it is sometimes difficult to list a priori all the possible hypotheses related to a given problem domain. He induced what he called the open-world assumption and the closed-world assumption (Smets, 1990) (Smets, 1998).

Under the open-world assumption, $\Theta$ is not necessarily exhaustive. It means that we admit that the problem domain can include some unknown hypotheses that we did not mention into the frame of discernment, whereas under the closed-world assumption the frame of discernment is exhaustive (on which our work is based).

## Example 1.1

Let us treat a problem of identification of childhood diseases. Some of the most common illnesses of childhood cause skin eruptions and are known as exanthems. The childhood exanthems include measles, rubella and fifth disease. All of these infections have the same symptoms.

Suppose the frame of discernment $\Theta$ related to this problem is defined as follows:

$$
\Theta=\{\text { measles, rubella, fifth disease }\}
$$

Then, the power set of $\Theta$ is:

$$
\begin{gathered}
2^{\Theta}=\{\emptyset,\{\text { measles }\},\{\text { rubella }\},\{\text { fifth disease }\}, \\
\{\text { measles, rubella }\},\{\text { measles, fifth disease }\},\{\text { rubella, fifth disease }\}, \Theta\}
\end{gathered}
$$

### 1.2.2 Basic belief assignment

The basic belief assignment (bba), called initially by Shafer basic probability assignment (Shafer, 1976), assigns a belief in range $[0,1]$ to every member of $2^{\Theta}$ (bba can assign belief to any proposition in the frame and not only to the elementary ones) such that their sum is 1 .
That means, a function is called a basic belief assignment such that (Shafer, 1976):

$$
\begin{equation*}
\sum_{A \subseteq \Theta} m(A)=1 \tag{1.2}
\end{equation*}
$$

The value $m(A)$, named a basic belief mass (bbm), represents the portion of belief committed exactly to the event $A$ and not for a particular subset of $A$. In this way, committing belief to a proposition $A$ does not necessarily imply that the remaining belief is committed to $\bar{A}$.

The mass $m(\Theta)$ quantifies the part of belief committed to the whole frame $\Theta$. It represents the beliefs that are not assigned to the different subsets of $\Theta$.

Similarly, $m(\emptyset)$ represents the part of belief allocated to the empty set. Shafer has initially imposed the condition $m(\emptyset)=0$. This condition reflects the fact that no belief ought to be allocated to the empty set. Such bba is called a normalized basic belief assignment.

However, this condition is relaxed in the TBM, the allocation of a positive mass to the empty set $(m(\emptyset)>0)$ is interpreted as a consequence of the openworld assumption (Smets, 1990). A mass of belief is assigned to each possible subset of classes.

## Example 1.2

Let us continue with Example 1.1. Suppose a doctor expressing a piece of evidence concerning the diseases.
The bba is then defined as follows:

$$
\begin{aligned}
& m(\{\text { measles }\})=0.6 ; \\
& m(\{\text { measles, rubella }\})=0.2 ; \\
& m(\Theta)=0.2
\end{aligned}
$$

For example, 0.6 represents the part of belief exactly committed to the hypothesis "the patient has measles".

### 1.2.3 Focal elements, body of evidence, core

The subsets $A$ of the frame of discernment $\Theta$ such that $m(A)$ is strictly positive, are called the focal elements of the bba $m$.

The pair $(F, m)$ is called a body of evidence where $F$ is the set of all the focal elements relative to the bba $m$.

The union of all the focal elements of $m$ are named the core and are defined as follows:

$$
\begin{equation*}
\varphi=\bigcup_{A: m(A)>0} A \tag{1.3}
\end{equation*}
$$

## Example 1.3

Let us continue with the Example 1.2, the subsets \{measles\}, \{measles, fifth disease $\}$, and $\Theta$ are the focal elements of the bba m.

So, $F=\{\{$ measles $\},\{$ measles, fifth disease $\}, \Theta\}$ is the set of the focal elements of $m$, and $(F, m)$ is called the body of evidence.

The core of this bba $m$ is defined as follows:
$\varphi=\{$ measles $\} \cup\{$ measles, fifth disease $\} \cup \Theta=\Theta$

### 1.2.4 Belief function

Belief function or credibility function, denoted bel, corresponding to a specific bba $m$, assigns to every subset $A$ of $\Theta$ the sum of the masses of belief committed exactly to every subset of $A$ by $m$ (Shafer, 1976).

Unlike the bbm $m(A)$ which measures the exact portion of belief assigned to the subset $A, \operatorname{bel}(A)$ quantifies the total amount of belief assigned to the subsets implying $A$ without implying $\bar{A}$. It is obtained by summing all the bbm's given to the subsets of $A$. Since $m(\emptyset)$ supports not only $A$, but also $\bar{A}$, the empty set must be discarded from the sum.

The belief function bel is defined for $A \subseteq \Theta$ and $A \neq \emptyset$ as:

$$
\begin{gathered}
\text { bel : : } 2^{\Theta} \rightarrow[0,1] \\
\operatorname{bel}(A)=\sum_{\emptyset \neq B \subseteq A} m(B)
\end{gathered}
$$

The belief function bel satisfies the following condition (Shafer, 1976): For all $A_{1}, \ldots, A_{n} \in 2^{\Theta}$,

$$
\begin{array}{r}
\operatorname{bel}\left(A_{1} \cup \ldots \cup A_{n}\right) \geq \sum_{i} \operatorname{bel}\left(A_{i}\right)- \\
\sum_{i>j} \operatorname{bel}\left(A_{i} \cap A_{j}\right)-\ldots-(-1)^{n} \operatorname{bel}\left(A_{i} \cap A_{n}\right) \tag{1.4}
\end{array}
$$

Shafer assumed that $\operatorname{bel}(\Theta)=1$ (Shafer, 1976). This can be ignored in the TBM, under the open world assumption, requiring only that $\operatorname{bel}(\Theta)<1$.

## Properties

- Sub-additivity:

$$
\begin{equation*}
\operatorname{bel}(A)+\operatorname{bel}(\bar{A}) \leq 1 \tag{1.5}
\end{equation*}
$$

Contrary to the probability theory the belief function theory increasing beliefs on a proposition A does not necessary require the decrease of beliefs on $\bar{A}$.

- Monotonicity:

$$
\begin{equation*}
A \subseteq B \Rightarrow \operatorname{bel}(B) \geq \operatorname{bel}(A) \tag{1.6}
\end{equation*}
$$

$\Theta$ will get the highest value of bel, whereas $\emptyset$ will get the lowest value.

- For $A, B \subseteq \Theta$ and $A \cap B=\emptyset$ :

$$
\begin{equation*}
\operatorname{bel}(A \cup B) \geq \operatorname{bel}(A)+\operatorname{bel}(B) \tag{1.7}
\end{equation*}
$$

- It is possible to obtain the basic belief assignment from the belief measure:

$$
\begin{equation*}
m(A)=\sum_{B \subseteq A}(-1)^{|A|-|B|} \operatorname{bel}(B), \forall A \subseteq \Theta, A \neq \emptyset \tag{1.8}
\end{equation*}
$$

- the bbm $m(\emptyset)$ is computed as follows:

$$
\begin{equation*}
m(\emptyset)=1-\operatorname{bel}(\Theta) \tag{1.9}
\end{equation*}
$$

## Example 1.4

Let us continue with Example 1.2. The belief function bel corresponding to the bba $m$ is defined by:

$$
\begin{aligned}
& \operatorname{bel}(\{\text { measles }\})=0.6 ; \\
& \operatorname{bel}(\{\text { rubella }\})=0 ; \\
& \operatorname{bel}(\{\text { fifth disease }\})=0 ; \\
& \operatorname{bel}(\{\text { measles, rubella }\})=0.8 ; \\
& \operatorname{bel}(\{\text { measles, fifth disease }\})=0.6 ; \\
& \operatorname{bel}(\{\text { rubella, fifth disease }\})=0 ; \\
& \operatorname{bel}(\Theta)=1 ;
\end{aligned}
$$

For example, 0.8 is the total amount of belief allocated to the proposition "measles, rubella" that is "the patient has the disease measles or rubella". It is obtained by summing the mass of this set with the masses of its subsets.

### 1.2.5 Plausibility function

The plausibility function $p l$, expresses the maximum amount of specific support that could be given to a proposition $A$ in $\Theta$. It measures the degree
of belief committed to the propositions compatible with $A . \operatorname{pl}(A)$ is then obtained by summing the bbm's given to the subsets $B$ such that $B \cap A \neq \emptyset$ (Shafer, 1976).

The plausibility function is defined by:

$$
p l(A)=\begin{gathered}
p l: 2^{\theta} \rightarrow[0,1] \\
\sum_{B \cap A \neq \emptyset} m(B), \quad \forall A \subseteq \Theta
\end{gathered}
$$

There is a simple relationship between the belief function bel and the plausibility function $p l$ associated with a mass function m: for $A \subseteq \Theta$

$$
\begin{equation*}
p l(A)=\operatorname{bel}(\Theta)-\operatorname{bel}(\bar{A}) \tag{1.10}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{bel}(A)=p l(\Theta)-\operatorname{bel}(\bar{A}) \tag{1.11}
\end{equation*}
$$

where $\bar{A}$ denotes the complement of A .

## Properties

- Over additivity:

$$
\begin{equation*}
p l(A)+p l(\bar{A}) \geq 1 \tag{1.12}
\end{equation*}
$$

- Monotonicity:

$$
\begin{equation*}
A \subseteq B \Rightarrow p l(B) \geq p l(A) \tag{1.13}
\end{equation*}
$$

- For $A, B \subseteq \Theta$ and $A \cap B=\emptyset$ :

$$
\begin{equation*}
p l(A \cup B) \leq p l(A)+p l(B) \tag{1.14}
\end{equation*}
$$

- For $A \subseteq \Theta$

$$
\begin{equation*}
\operatorname{bel}(A) \leq p l(A) \tag{1.15}
\end{equation*}
$$

The two measures, belief and plausibility, can be derived from each other:

$$
\begin{equation*}
p l(A)=1-\operatorname{bel}(\bar{A}) \tag{1.16}
\end{equation*}
$$

Otherwise, $\operatorname{bel}(A)$ and $p l(A)$ may be viewed as lower and upper bounds on probabilities $(\operatorname{bel}(A) \leq p l(A))($ Smets, 1990).

## Example 1.5

Let us continue with Example 1.3. The plausibility function pl corresponding to the bba $m$ is defined by:
$p l(\{$ measles $\})=1 ;$
$p l(\{$ rubella $\})=0.4 ;$
$p l(\{$ fifth disease $\})=0.2$;
$p l(\{$ measles, rubella $\})=1 ;$
$p l(\{$ measles, fifth disease $\})=1$;
$p l(\{$ rubella, fifth disease $\})=0.4 ;$
$p l(\Theta)=1 ;$
For example, 0.4 represents the maximum degree of belief that could be given to the the proposition"rubella".

## Remarks

- There is another function used to simplify computations like the commonality function (Barnett, 1991). It is defined as follows:

$$
q(A)=\begin{gathered}
q: 2^{\theta} \rightarrow[0,1] \\
\sum_{A \subseteq B} m(B), \quad \forall A \subseteq \Theta
\end{gathered}
$$

However, it may represent the total mass that is free to move to every elements of A.

- The basic belief assignment, the belief function, the plausibility function and the commonality function are considered as different expressions of the same information.
- The quantity $p l(A)$ is seen as the degree of maximal (or potential) support attributed to the proposition $A$, whereas $\operatorname{bel}(A)$ is seen as the degree of minimal (or necessary) support attributed to $A$. This information may be conveniently expressed by the interval $[\operatorname{bel}(A), p l(A)]$, called belief interval (Smets, 1990).


### 1.3 Special belief functions

In this Section, we present some special belief functions relative to particular states of uncertainty.

### 1.3.1 Vacuous belief function

A vacuous belief function is a normalized belief function with $\Theta$ is its unique focal element (Shafer, 1976). So, its corresponding bba is defined as follows:

$$
\begin{equation*}
m(\Theta)=1 \text { and } m(A)=0, \quad \forall A \subset \Theta, A \neq \Theta \tag{1.17}
\end{equation*}
$$

In other words,

$$
\begin{equation*}
\operatorname{bel}(\Theta)=1 \operatorname{and} \operatorname{bel}(A)=0, \quad \forall A \subset \Theta, A \neq \Theta \tag{1.18}
\end{equation*}
$$

Such basic belief assignment quantifies the state of total ignorance, in which there is no reason to belief in any proposition more than another and all the propositions are plausible.

## Example 1.6

Suppose that a doctor cannot detect the nature of disease. We have a state of total ignorance where the corresponding bba is a vacuous bba defined by:

$$
m(\Theta)=1 \text { and } m(A)=0, \quad \forall A \neq \emptyset
$$

### 1.3.2 Categorical belief function

It is a normalized belief function such that its bba is defined as follows (Mellouli, 1987):

$$
\begin{equation*}
m(A)=1, \quad \text { for some } A \subset \Theta \tag{1.19}
\end{equation*}
$$

and

$$
\begin{equation*}
m(B)=0, \quad \forall B \subseteq \Theta, \quad B \neq A \tag{1.20}
\end{equation*}
$$

Such function has a unique focal element A (which is not imperatively a singleton event) different from the frame of discernment $\Theta$.

## Example 1.7

Suppose we get a piece of evidence assuring that the disease nature cannot be rubella. The corresponding bba is a categorical belief function such that:

$$
m(\{\text { measles, fifth disease }\})=1
$$

### 1.3.3 Certain belief function

A certain belief function is a categorical belief function such that its focal element is a singleton. It represents a state of total certainty:

$$
\begin{equation*}
m(A)=1, \quad \text { for some } A \subset \Theta, \quad|A|=1 \tag{1.21}
\end{equation*}
$$

and

$$
\begin{equation*}
m(B)=0, \quad \forall B \subseteq \Theta, \quad B \neq A \tag{1.22}
\end{equation*}
$$

Such function represents a state of total certainty as it assigns all the belief to a unique elementary event.

## Example 1.8

Let's consider the doctor's confirmation that the disease is "measles". The bba corresponding is a certain bba defined as:

$$
m(\{\text { measles }\})=1
$$

### 1.3.4 Bayesian belief function

A bayesian belief function is a special belief function assigning non-zero masses to singletons only (Shafer, 1976):

$$
\begin{equation*}
m_{b}(A)=0, \quad|A|>1 \tag{1.23}
\end{equation*}
$$

$m_{b}$ becomes a probability distribution.

## Example 1.9

Assume a piece of evidence expressed by the following bba:
$m_{b}(\{$ measles $\})=0.2$,
$m_{b}(\{$ rubella $\})=0.7$;
$m_{b}(\{$ fifth disease $\})=0.1$;
$m_{b}(\Theta)=0 ;$
$m_{b}$ is a Bayesian bba since all its focal elements are singletons.

### 1.3.5 Simple support function

A belief function is called a simple support function (ssf) if it has at most one focal element different from the frame of discernment $\Theta$. This focal element is called the focus of the ssf.

A simple support function is defned as follows (Smets, 1995):

$$
m(X)=\left\{\begin{array}{l}
w \text { if } X=\Theta  \tag{1.24}\\
1-w \text { if } X=A \text { for some } A \subset \Theta \\
0 \text { otherwise }
\end{array}\right.
$$

where $A$ is the focus of the ssf and $w \in[0,1]$.

## Example 1.10

Let us continue with the Example 1.1. Assume we have a bba defined as follows:
$m(\{$ measles, rubella $\})=0.7 ;$
$m(\Theta)=0.3 ;$
$m$ is called a simple support function where the focus is the proposition $\{$ measles, rubella $\}$.

### 1.3.6 Consonant belief function

A consonant belief function is a belief function in which all the focal elements ( $A_{1}, A_{2}, \ldots, A_{n}$ ) are nested, that is $A_{1} \subseteq A_{2} \subseteq \ldots \subseteq A_{n}$.

## Properties

- bel is a necessity measure:

$$
\begin{equation*}
\operatorname{bel}(A \cap B)=\min (\operatorname{bel}(A), \operatorname{bel}(B)) \tag{1.25}
\end{equation*}
$$

- $p l$ is a possibility measure:

$$
\begin{equation*}
p l(A \cup B)=\max (p l(A), p l(B)) \tag{1.26}
\end{equation*}
$$

## Example 1.11

Let us consider the same bba defined in Example 1.1:
$m(\{$ measles $\})=0.7 ;$
$m(\{$ measles, fifth disease $\})=0.3$;
$m(\Theta)=0.2 ;$

### 1.3.7 Dogmatic and non-dogmatic belief functions

A belief function is said to be dogmatic if and only if its corresponding bba $m$ is such that $m(\Theta)=0$. This case involves some previous cases (certain belief functions, Bayesian belief functions, categorical belief functions). A non-dogmatic belief function is defined such that $m(\Theta)>0$ (Smets, 1995).

### 1.4 Combination

The belief function theory, as understood in the TBM framework, is a mathematical theory that offers interesting tools for aggregating the basic belief assignments defined over the same frame of discernment and induced from distinct pieces of evidence and provided by two (or more) source of
information. Its purpose is then to summarize and to simplify a corpus of data.

### 1.4.1 Combination of two information sources

Let $m_{1}$ and $m_{2}$ be two bba's induced from two distinct information sources and defined on the same frame of discernment $\Theta$. The combination of these bba's induces a bba on the same frame $\Theta$. The combination can be either conjunctive or disjunctive.

The choice of one of these rules of combination for aggregating pieces of evidence may be guided by meta-belief concerning the reliability of the sources. In fact, if we know that both sources of information are fully reliable, then we combine them conjunctively. However, if we know that at least one of the two sources is reliable, then we combine them disjunctively (Smets, 1990) (Smets, 1991).

## Conjunctive rule of combination

When we know that both sources of information are fully reliable, the resulting bba is computed by the conjunctive rule of combination. Hence, the induced bba quantifies the combined impact of the two pieces of evidence. It is defined as follows (Smets, 1990):

$$
\begin{equation*}
\left(m_{1} @ m_{2}\right)(A)=\sum_{B, C \subseteq \Theta, B \cap C=A} m_{1}(B) m_{2}(C), \quad \forall A \subseteq \Theta \tag{1.27}
\end{equation*}
$$

The conjunctive rule is considered as the unnormalized Demspter's rule of combination dealing with the closed world assumptions, defined as follows (Shafer, 1986):

$$
\begin{equation*}
\left(m_{1} \oplus m_{2}\right)(A)=K \cdot \sum_{B, C \subseteq \Theta, B \cap C=A} m_{1}(B) m_{2}(C) \tag{1.28}
\end{equation*}
$$

where

$$
\begin{equation*}
K^{-1}=1-\sum_{B, C \subseteq \Theta, B \cap C=\emptyset} m_{1}(B) m_{2}(C) \tag{1.29}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(m_{1} \oplus m_{2}\right)(\emptyset)=0 \tag{1.30}
\end{equation*}
$$

$K$ is called the normalization factor.

## Properties

The conjunctive rule of combination is characterized by the following properties:

- Commutativity:

$$
\begin{equation*}
m_{1} ® m_{2}=m_{2} @ m_{1} \tag{1.31}
\end{equation*}
$$

- Associativity:

$$
\begin{equation*}
\left(m_{1} @ m_{2}\right) ® m_{3}=m_{1} @\left(m_{2} @ m_{3}\right) \tag{1.32}
\end{equation*}
$$

- Non-idempotency:

$$
\begin{equation*}
m @ m \neq m \tag{1.33}
\end{equation*}
$$

- Neutral element:

The neutral element within the conjunctive rule of combination is the vacuous basic belief assignment representing the total ignorance:

$$
\begin{equation*}
m ® m_{0}=m \tag{1.34}
\end{equation*}
$$

where $m_{0}$ is a vacuous bba.

## Example 1.12

Let us consider two distinct doctors' evidences $S_{1}$ and $S_{2}$. The first evidence is expressed by a bba $m_{1}$ and defined as follows:
$m_{1}(\{$ measles $\})=0.4 ;$
$m_{1}(\{$ rubella $\})=0.1$;
$m_{1}(\{$ measles, fifth disease $\})=0.3 ;$
$m_{1}(\Theta)=0.2 ;$
The second evidence is expressed by $m_{2}$ defined as follows:
$m_{2}(\{$ fifth disease $\})=0.5$;
$m_{2}(\{$ measles, fifth disease $\})=0.4$;
$m_{2}(\Theta)=0.1 ;$

The bba corresponding to the conjunctive combination of both pieces of evidence is defined as follows:

```
\(\left(m_{1} @ m_{2}\right)(\emptyset)=0.2+0.05+0.04=0.29 ;\)
\(\left(m_{1} \bigcirc m_{2}\right)(\{\) measles \(\})=0.16+0.04=0.2 ;\)
\(\left(m_{1} \bigcirc m_{2}\right)(\{\) rubella \(\})=0.01\);
\(\left(m_{1} @ m_{2}\right)(\{\) fifth disease \(\})=0.1+0.15=0.25\);
\(\left(m_{1} \oslash m_{2}\right)(\{\) measles, fifth disease \(\})=0.08+0.12+0.02=0.22\);
\(\left(m_{1} @ m_{2}\right)(\Theta)=0.03 ;\)
```

$m_{1} ® m_{2}$ represents the joint bba induced from the combination of $m_{1}$ and $m_{2}$ by using the conjunctive rule of combination.

### 1.4.2 Disjunctive rule of combination

The dual of the conjunctive rule is the disjunctive rule of combination. We use it when we only know that at least one of the sources of information is reliable but we do not know which one is reliable (Smets, 1998).

$$
\begin{equation*}
\left(m_{1} \circlearrowleft m_{2}\right)(A)=\sum_{B, C \subseteq \Theta, B \cup C=A} m_{1}(B) m_{2}(C) \tag{1.35}
\end{equation*}
$$

The disjunctive rule of combination (as the conjunctive rule of combination) is commutative and associative.

## Example 1.13

Let us consider the same bbas represented in Example 1.10. Once the disjunctive rule of combination is applied, we get:
$\left(m_{1} \circlearrowleft m_{2}\right)(\{$ measles, fifth disease $\})=0.2+0.1+0.16+0.08=0.54 ;$
$\left(m_{1} @ m_{2}\right)(\{$ rubella, fifth disease $\})=0.05$;
$\left(m_{1} \circlearrowleft m_{2}\right)(\Theta)=0.15+0.12+0.04+0.02+0.03+0.04+0.01=0.41 ;$
$m_{1} @ m_{2}$ represents the joint bba induced from the combination of $m_{1}$ and $m_{2}$ by using the disjunctive rule of combination.

### 1.4.3 Combination of several information sources

Since the conjunctive and the disjunctive rules of combination are both commutative and associative, combining several pieces of evidence induced from distinct information sources (either conjunctively or disjunctively) may be easily ensured by applying repeatedly the chosen rule.

## Remark

In addition to the conjunctive and disjunctive combination rule, a larger choice of combination rules has been recognized by many researchers involved in real-world applications (Lefevre et al., 2002) (Yager, 1987): Yagers rule (Yager, 1987), the cautious rule (Denoeux, 2006), and Inagakis unified combination rule (Inagaki, 1991), etc.

### 1.5 Generalized bayes theorem

Smets has Generalized the Bayesian Theorem (GBT) (Smets, 1991), offering an interesting tool for inverting conditional belief functions within the TBM framework. Assume that we have a vacuous a priori belief on a frame $\Theta$ (that is we are in state of total ignorance), and we know for each element $\theta \in \Theta$, what would be our belief on another frame $X$, if this element happened.

Suppose that we learn that the actual value of $X$ is in $x \in X$, then the GBT allows us to derive the conditional belief function over the frame $\Theta$ given this observation. One has (Smets, 1991):

$$
\begin{equation*}
p l^{\Theta}[x](\theta)=1-\prod_{\theta_{i} \in \Theta}\left(1-p l^{X}\left[\theta_{i}\right](x)\right) \tag{1.36}
\end{equation*}
$$

Furthermore, if we assume we have only some beliefs on the value of $x$, and these beliefs are represented by a belief function bel ${ }^{X}$ over $X\left(m^{X}\right.$ its bba), then the GBT becomes:

$$
\begin{equation*}
p l^{\Theta}\left[m^{X}\right](\theta)=\sum_{x \subseteq X} m^{X}(x) p l^{\Theta}[x](\theta) \tag{1.37}
\end{equation*}
$$

where

$$
\begin{equation*}
p l^{\Theta}[\emptyset](\theta)=0 \tag{1.38}
\end{equation*}
$$

The bba's and the belief functions on $\Theta$ are computed from these plausibility functions.

### 1.6 Discounting

In the Transferable Belief Model, discounting allows to take in consideration the reliability of the information source that generates the bba $m$.

For $\alpha \in[0,1]$, let (1- $\alpha$ ) be the degree of confidence ('reliability') we assign to the source of information. It quantifies the strength of reliability given to the expert (Smets, 1992).

If the source is not fully reliable, the expert's opinions are represented as follows:

$$
\begin{gather*}
m^{\alpha}(A)=(1-\alpha) m(A), \quad \text { for } A \subset \Theta  \tag{1.39}\\
m^{\alpha}(\Theta)=\alpha+(1-\alpha) m(\Theta) \tag{1.40}
\end{gather*}
$$

Where $\alpha$ is the discounting factor.

## Properties

- $\alpha=0$ means that the expert is totally reliable. So, the discounting does not affect the bba m. That's:

$$
\begin{equation*}
m^{\alpha}=m \tag{1.41}
\end{equation*}
$$

- $\alpha=1$ means that the expert is not reliable at all. Then, his opinions have to be totally ignored. Thus, the bba $m$ is reduced to a vacuous belief function.


## Example 1.14

The degree of reliability given to the expert is equal to 0.7. If we consider this bba defined as follows:
$m(\{$ measles $\})=0.4 ;$
$m(\{$ measles,fifth disease $\})=0.3 ;$
$m(\Theta)=0.3 ;$
So, we obtain after discounting this bba:
$m^{\alpha}(\{$ measles $\})=0.4 \times 0.7=0.28 ;$
$m^{\alpha}(\{$ measles, fifth disease $\})=0.3 \times 0.7=0.21$;
$m^{\alpha}(\Theta)=0.3+(0.3 \times 0.7)=0.51 ;$

### 1.7 Coarsening and refinement

Sometimes, beliefs are induced by information sources with different but compatible frames of discernment. The coarsening and refinement operations allow to establish relationships between these different frames in order to express beliefs on anyone of them.

### 1.7.1 Refinement and coarsening

Let $\Omega$ and $\Theta$ be two finite sets. The idea behind the refinement consists in obtaining one frame of discernment $\alpha$ from the set $\Theta$ by splitting some or all of its events (Shafer, 1976).

Conversely, the coarsening consists in forming a frame $\Theta$ by grouping together the events of the frame of discernment $\Omega$.

Let us define a mapping $\rho: 2^{\Theta} \rightarrow 2^{\Omega}$ such that (Shafer, 1976):

$$
\begin{gather*}
\rho(\{\theta\}) \neq \emptyset \quad \forall \theta \in \Theta  \tag{1.42}\\
\rho(\{\theta\}) \cap \rho\left(\left\{\theta^{\prime}\right\}\right)=\emptyset \text { if } \theta \neq \theta^{\prime}  \tag{1.43}\\
\bigcup_{\theta \in \Theta} \rho(\{\theta\})=\Omega \tag{1.44}
\end{gather*}
$$

So, given a disjoint partition $\rho(\{\theta\})$ one may set (Shafer, 1976):

$$
\begin{equation*}
\rho(A)=\bigcup_{\theta \in A} \rho(\{\theta\}) \tag{1.45}
\end{equation*}
$$

For each $A \in \Theta, \rho(A)$ consists of all the possibilities in $\Omega$ by splitting the elements of $A$ (Shafer, 1976).

The mapping $\rho$ : $2^{\Theta} \rightarrow 2^{\Omega}$ is called a refining, $\Omega$ is a refinement of $\Theta$ and $\Theta$ is the coarsening of $\Omega$.

## Example 1.15

Let us continue with the same problem domain.

$$
\begin{equation*}
\Theta=\{\text { measles, rubella, fifth disease }\} \tag{1.46}
\end{equation*}
$$

A possible refinement of the frame of discernment $\Theta$ is:
$\Omega=\{$ childhood_measles, congenital_measles, childhood_rubella, congenital_rubella,
childhood_fifth disease, congenital_fifth disease $\}$
where
$\rho($ measles $)=\{$ childhood_measles, congenital_rubella $\}$
$\rho($ rubella $)=\{$ childhood_rubella, congenital_fifth disease $\}$
$\rho($ fifth disease $)=\{$ childhood_fifth disease, congenital_fifth disease $\}$
Inversely, $\Theta$ is considered as the coarsening of $\Omega$.

### 1.7.2 Definition of bba's

Due to refinement and regarding bba's, it is easy to update a bba $m^{\Theta}$ defined on the frame of discernment $\Theta$ to a refinement $\Omega$, we get the bba $m^{\Omega}$ defined as follows:

$$
m^{\Theta}(B)=\left\{\begin{array}{l}
m^{\Omega}(A) \text { if } B=\rho(A) \text { for some } A \subseteq \Theta  \tag{1.47}\\
0 \text { otherwise }
\end{array}\right.
$$

### 1.8 Decision making

It is necessary when making a decision, to select the most likely hypothesis. Some solutions are developed to ensure the decision making within the belief function theory. One of the most used is the pignistic probability proposed within the TBM (Smets \& Kennes, 1994).

The TBM is a model whose aims at quantifying someone's degree of belief. It is based on a two level (Smets \& Kennes, 1994):

- The credal level where beliefs are entertained and represented by belief functions.
- The pignistic level where beliefs are used to make decisions and represented by probability functions called the pignistic probabilities.

When a decision must be made, beliefs held at the credal level induce a probability measure at the pignistic measure denoted BetP (Smets, 1998):

$$
\begin{equation*}
\operatorname{Bet} P(A)=\sum_{B \subseteq \Theta} \frac{|A \cap B|}{|B|} \frac{m(B)}{(1-m(\emptyset))}, \quad \forall A \in \Theta \tag{1.48}
\end{equation*}
$$

It includes normalization and division of bba's assigned to focal elements by their cardinality.

## Example 1.16

Assume that at the credal level, beliefs are represented by the following bba:

```
\(m(\{\) measles \(\})=0.7 ;\)
\(m(\{\) measles, rubella \(\})=0.2 ;\)
\(m(\Theta)=0.1 ;\)
```

To select the most probably hypothesis, we have to compute the corresponding pignistic probabilities BetP to make the optimal decision, so we have:
$\operatorname{Bet} P(\{$ measles $\})=0.84 ;$
$\operatorname{Bet} P(\{$ rubella $\})=0.13$;
$\operatorname{Bet} P(\{$ fifth disease $\})=0.03$;
We notice that the most probable disease is "measles". So, if we have to decide, we will choose this hypothesis.

We mention other methods like the maximum of credibility which consists in choosing the hypothesis having the highest value of the belief function bel, that is the most credible hypothesis and the maximum of plausibility,
contrary to the maximum credibility criterion, this method consists in supporting the hypothesis having the highest value of the plausibility function (Barnett, 1991).

### 1.9 Conclusion

In this chapter, we have presented the basic concepts of belief function theory as understood in the Transferable Belief Model.

This presentation shows that the belief function theory provides a convenient tool to handle uncertainty in decision problems, especially within Multi-Criteria Decision Making Methods. The following chapter will deal with these methods more precisely the Analytic Hierarchy Process.

## Chapter 2

## Multi-Criteria Decision Making

### 2.1 Introduction

Within the framework of Multi-Criteria Decision Making (MCDM) problems, a decision maker often needs to make judgments on decision alternatives that are evaluated on the basis of its preferences (criteria) (Zeleny, 1982). However, this is not an easy task because often these criteria may be conflicting with each other.

As a result, a number of MCDM methods were proposed, and each one has its own characteristics. Amongst the most well known ones is the Analytic Hierarchy Process (AHP) (Saaty, 1977) (Saaty, 1980).

In this chapter, we firstly present an overview of MDCM: we will briefly introduce the basic concepts. Then, we expose several MCDM methods, organized into two major families: Multi-attribute Utility Theory (MAUT) and outranking methods. We are interested especially in AHP method: we focus on its standard version where its procedure will be described, then an example will be detailed to illustrate this approach.

The last part of this chapter deals with another kind of this approach under uncertain environment which is briefly exposed combining this method with one theory managing this kind of environment such as fuzzy theory, probability theory and belief function theory.

### 2.2 What is Multi-Criteria Decision Making?

### 2.2.1 Definition

Our life is filled with many decisions: from the simple everyday problem of selecting a school to the complex problems of economic planning. Real-life decisions involve multiple criteria that are most likely conflicting with each other (Zeleny, 1982).

The mathematical representation of these decision making problems started in the $19^{\text {th }}$ century with economists and applied mathematicians like Pareto, VonNeumann, Morgenstern, etc. The first approaches considered monocriterion decision problems, and in 1951, the multi-criteria problem was introduced by Koopmans, Kuhn and Tucker (Figueira et al., 2005).

By definition, MCDM will allow the decision maker to determine which the best alternatives are, considering multiple conflicting criteria or goals. Its general purpose is to serve as an aid to thinking and decision making. That's why, the concept of optimum does not exist in a multicriteria framework.

In the MCDM field, three kinds of problems are distinguished (Roy, 1996): choice problems $\left(P_{\alpha}\right)$, ranking problems $\left(P_{\beta}\right)$ and sorting problems $\left(P_{\mu}\right)$. The goal of the decision maker in each type of problem is different: in choice problems, the aim is to find the best alternative. In ranking problems, we want to know the goodness of all alternatives, which is usually presented as a ranking from the best to the worst, and in sorting problems we want to know which alternatives belong to each class of a predefined set of classes.

### 2.2.2 Basic concepts

Although MCDM problems could be very different in context, they share the following common features. There are the notions of alternatives, criteria, etc. In this section, we define the different concepts (Triantaphyllou, 2000) (Figueira et al., 2005):

- Decision maker: Actor for whom the decision-aid tools are developed and implemented.
- Alternative: Usually alternatives represent the different choices of action available to the decision maker.
- Attribute: Attributes are also referred to as "goals" or "decision criteria". Attributes represent the different dimensions from which the alternatives can be viewed.
We assume an application $g$, such that it appears meaningful to compare two alternatives $a_{1}$ and $a_{2}$ according to a particular point of view, on the sole basis of $g\left(a_{1}\right)$ and $g\left(a_{2}\right)$.
We will follow $C=\left\{c_{1}, c_{2}, \ldots, c_{m}\right\}$ to denote the criteria, being $g_{j}$ the function attached to $c_{j}$. Besides with $v_{i j}$, we denote the value of $g_{j}\left(a_{i}\right)$. That is $v_{i j}=g_{j}\left(a_{i}\right)$.
Criteria can be both well defined and quantitatively measurable (price, size, etc.) or qualitative and difficult to measure (appearance, satisfaction, etc.). It should be:
- able to discriminate among the alternatives and to support the comparison of the performance of the alternatives,
- complete to include all goals,
- operational and meaningful,
- non-redundant,
- few in number.
- Weight: Value that indicates the relative importance of one criterion in a particular decision process (denoted by $w$ ). These weights are usually normalized.
- Performance matrix: Consider a MCDM problem with $m$ criteria and $n$ alternatives. Let $c_{1}, \ldots, c_{m}$ and $a_{1}, \ldots, a_{n}$ denote the criteria and alternatives, respectively. A standard feature of MCDM methodology is the decision table as shown below (see Table 2.1).
In the table each column belongs to a criterion and each row describes the performance of an alternative. The score $v_{i j}$ describes the performance of alternative $a_{j}$ against criterion $c_{i}$.

Table 2.1: Decision matrix

|  | Criteria |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| Alternatives | $c_{1}$ | $c_{2}$ | $\ldots$ | $c_{m}$ |
| $a_{1}$ | $v_{11}$ | $v_{12}$ | $\ldots$ | $v_{1 m}$ |
| $\ldots$ |  |  |  |  |
| $a_{n}$ |  |  |  |  |

### 2.3 Decision making process

Whether simple or complex, all decisions involve the same basic process (Belton \& Stewart, 2003):

1. Establish aims of the MCDM, and identify decision makers and other key players.
2. Identify the alternatives to be appraised.
3. Identify objectives and criteria.

- Identify criteria for assessing the consequences of each alternative.
- Organise the criteria by clustering them under high-level and lowerlevel objectives in a hierarchy.

4. Assign weights for each of the criterion to reflect their relative importance to the decision.
5. Select a decision making tool.
6. The alternatives are then evaluated using key available information and the set of established criteria.
7. Finally, a decision is made regarding the best alternative, and probable consequences of this decision are assessed. It may be necessary to repeat the whole process if it is found that a misjudgment has been made.

### 2.4 Multi-Criteria Decision Making methods

A large number of MCDM method exist in the literature and there are many ways to classify them. One way is to classify them according to how they process the basic information in the performance matrix. That means, the principal difference between the main families of MCDM methods is the way in which this aggregation is done. That's why these methodologies are classified into two distinct families (Figueira et al., 2005) (Triantaphyllou, 2000): On the one hand, the outranking approach introduced by Roy where some methods like Electre and Promothee are developed (Brans et al., 1986) (Figueira et al., 2005). On the other hand, the value and utility theory
approaches mainly started by Keeney and Raiffa (Keeney \& Raiffa, 1976), and then implemented in a number of methods (Triantaphyllou, 2000).

### 2.4.1 Outranking methods

The Outranking methods, proposed by Roy (1968) (Zeleny, 1982) (Figueira et al., 2005), are one of the most known approach of MCDM.

The basic idea is that small differences between alternatives are indifferent, and differences over some certain magnitude do not bring any additional value. For example, when buying a car, it does not make a difference for most of the decision makers whether the car costs 10000 euros or 20 euros more. In analogy, if one car costs 10000 and two others 2000000 and 3000000 , probably there is no difference between preferability of the first over the second one to the first over the third one. Both of the latter ones are considered "bad" with respect to the price of the first one.

Outranking methods are called such, because instead of aggregating their criterion values to a single attribute describing goodness of the alternative, they form an outranking relation between alternatives. An alternative is said to outrank another if it is considered as good as or better. In fact, their main idea is to establish preference ordering of alternatives by comparing all feasible alternatives or actions by pairs. Then, a concordance relation is established by aggregating the relative preferences. After having determined for each pair of alternatives whether one alternative outranks another, these pair-wise outranking assessments can be combined into a partial or complete ranking.

The basis of these methods is the definition of an outranking relation $S$. By definition, $S$ is a binary relation: $a^{\prime} S a$ holds that's " $a^{\prime}$ outranks alternative $a$ ". If given the information about the preferences of the decision maker, there are enough arguments to confirm that " $a$ " is at least as good as $a "$, and there is really no important reason to refuse this statement.

For alternative pairs $a^{\prime}$ and $a$, preferences are expressed for each criterion as one of the following types (Linkov et al., 2004):

- $a^{\prime} P a$, strict preference $a^{\prime}$ over $a$.
- $a^{\prime} Q a$, weak preference for $a^{\prime}$ over $a$.
- $a^{\prime} I a$, indifference between the two actions.
- $a^{\prime} J a$, inability or refusal to compare the actions.

The indifference threshold is the difference beneath which a decisionmaker has no preference: that is, a difference that is too small to be used as a basis of distinction between the two. The preference threshold is the difference above which the decision maker strongly prefers one management alternative to another.

Two conditions must be fulfilled in order to accept that $a^{\prime} S a$ holds:

1. A concordance condition: a majority of criteria must support a'Sa (classical majority principle).
2. A non discordance condition: among the non concordant criteria, none of them strongly refutes $a^{\prime} S a$ (respect of minorities principle)

The two most popular families of the outranking methods are: Electre (Figueira et al., 2005) and Promothee (Figueira et al., 2005) (Brans et al., 1986) methods.

### 2.4.2 Multi-Attribute Utility Theory

The multi-attribute utility theory (MAUT) is one of the oldest and well established MCDM theory. It was introduced by Keeney and Raiffa (1976).

The basis of MAUT is the use of utility functions. Its role is to transform diverse criteria into one common scale of utility or value. These values describe the "goodness" of alternatives taking into account the preferences of the decision maker. The alternative with the highest expected utility is the most preferred one, or "best" in the considered problem setting. The goal of decision maker is then to maximize the utility function. In other term, to maximize some function that aggregates the utility of each different criterion.

$$
\begin{equation*}
U=U\left(c_{1}, c_{2}, \ldots, c_{m}\right) \tag{2.1}
\end{equation*}
$$

Different models exist according to different expressions for function $U$ : the additive model (Triantaphyllou, 2000), the multiplicative model (Triantaphyllou, 2000), etc.

Another approach based on the MAUT principles is the Analytic Hierarchy Process (Saaty, 1977) (Saaty, 1980) where the problem is structured hierarchically. The purpose of constructing the hierarchy is to evaluate the influence of the criteria on the alternatives to attain objectives. This method was extended to Analytic Network Process (ANP) (Saaty, 1996), a generalization of the AHP method.

## Remark:

The choice of MCDM method depends not only on the criteria and the preferences of the decision maker, but also on the type of the problem. Therefore, for all the methods applied, the analyst as well as the decision maker should acknowledge the prerequisites for its use, as well as the advantages and disadvantages the method has.

### 2.5 Analytic Hierarchy Process as a MCDM method

The Analytic Hierarchy Process has been developed by (Saaty, 1977) (Saaty, 1980) and is one of the best known and most widely used MCDM approaches.

The AHP has attracted the interest of many researchers because it provides a flexible and easily understood way to analyze and decompose the complex decision problem through breaking it into smaller and smaller parts. In addition, It is a MCDM methodology that allows subjective as well as objective factors to be considered in the evaluation process. The pertinent data are then derived by using a set of pair-wise comparisons. These comparisons are used to obtain the weights of importance of the decision criteria, and the relative performance measures of the alternatives in terms of each individual decision criterion.

Indeed, that is the reason why AHP has successfully been applied to many practical problems (Saaty, 1990): from the simple problem of buying a car
to the complex problems of economic planning, portfolio selection, ressource allocation, etc.

The AHP, as a compensatory method, assumes complete aggregation among criteria and develops a linear additive model. The weights and scores are achieved basically by pair-wise comparisons between all alternatives and criteria. The basic procedure to carry out the AHP method will be presented in the following subsections.

### 2.5.1 Constructing the hierarchy

Constructing the hierarchical structure is the most important step in AHP method. In fact, this approach requires the decision maker to represent the problem within a hierarchical structure. The purpose of constructing the hierarchy is to evaluate the influence of the criteria on the alternatives to attain objectives.

The number of levels depends upon the complexity of the problem and the degree of detail in the problem. So, an AHP hierarchy has at least three levels: the main objective of the problem is represented at the top level of the hierarchy. Then each level of the hierarchy contains criteria or subcriteria that influence the decision. The last level of the structure contains the alternatives.

### 2.5.2 Pair-wise comparison

In AHP, once the hierarchy has been constructed, the decision maker begins the prioritization procedure to determine the relative importance of the elements on each level of the hierarchy (criteria and alternatives). Elements of a problem on each level are paired (with respect to their common relative impacts on a property or criteria) and then compared.

To compare elements on each level of the hierarchy, AHP uses a quantitative comparison method that is based on pair-wise comparisons of the following type "How important is criterion $c_{i}$ relative to criterion $c_{j}$ ?" Questions of this type are used to establish the weights for criteria and similar questions are used to assess the performance scores for alternatives on the subjective (judgmental) criteria. "How important is alternative $A$ when compared to alternative $B$ with respect to a specific criterion $c_{j}$ (in the level immediately

## higher)?"

The responses to the pair-wise comparison question use the following nine-point scale (Saaty scale) Table 2.2 expressing the intensity of the preference for one element versus another.

Table 2.2: The Saaty Rating Scale

| Intensity of <br> importance | Definition | Explanation |
| :--- | :--- | :--- |
| 1 | Equal impor- <br> tance | Two factors contribute equally to the objective. |
| 3 | Somewhat more <br> important | Experience and judgement slightly favour one over <br> the other. |
| 5 | Much more im- <br> portant | Experience and judgement strongly favour one <br> over the other. |
| 7 | Very much more <br> important | Experience and judgement very strongly favour <br> one over the other. Its importance is demonstrated <br> in practice. |
| 9 | Absolutely more <br> important. | The evidence favouring one over the other is of the <br> highest possible validity. |
| $2,4,6,8$ | Intermediate <br> values | When compromise is needed. |

Let $c_{i j}$ denote the value obtained by comparing criterion $c_{i}$ relative to criterion $c_{j}$. Of course, we set $c_{i i}=1$. Furthermore, if we set $c_{i j}=k$, then we set $c_{j i}=\frac{1}{k}$. For example, if criterion $c_{i}$ is absolutely more important than criterion $c_{j}$ and is rated at 9 , then $c_{j}$ must be absolutely less important than $c_{i}$ and is valued at $\frac{1}{9}$. Next, the comparison matrix is formed by repeating the process for each criterion.

Extracting the judgments enables the construction of the matrix of A $(n \times n)$, where $n$ elements compared to each other with respect to a specific criterion $c_{i}$.

### 2.5.3 Priority vector

After filling the pair-wise comparison matrices according to the 1-9 scale, the local priority weights are determined by using the eigenvalue method. The objective is then to find the weight of each element, or the score of each alternative by calculating the eigenvalue vector.

The formulation used in this method is shown below:

$$
\begin{equation*}
A w=\lambda_{\max } w \tag{2.2}
\end{equation*}
$$

Where $w$ is the right eigenvector and $\lambda_{\max }$ the maximum eigenvalue for the comparison matrix $A$.

### 2.5.4 Consistency ratio

In decision-making, it is important to know how good the consistency is. Consistency in this case means that the decision procedure is producing coherent judgments in specifying the pair-wise comparison of the criteria or alternatives.

However, perfect consistency rarely occurs in practice. In the AHP the pair-wise comparisons in a judgment matrix are considered to be adequately consistent if the corresponding consistency ratio $(C R)$ is less than $10 \%$. The CR coefficient is calculated as follows.

First, the consistency index ( $C I$ ) needs to be estimated. This is an index to assess how much the consistency of pair-wise comparison differs from perfect consistency. This is done by:

$$
\begin{equation*}
C I=\left(\lambda_{\max }-n\right) /(n-1) \tag{2.3}
\end{equation*}
$$

where $n$ is the matrix size and $\lambda_{\text {max }}$ the maximum eigenvalue.
Then, AHP measures the overall consistency of judgment by means of consistency ratio CR. The CR index is obtained by dividing the computed CI index by a random index (RI). Table 2.3, derived from Saaty's book, shows the RI for matrices of order 1 through 10 .

$$
\begin{equation*}
C R=\frac{C I}{R I} \tag{2.4}
\end{equation*}
$$

Table 2.3: Average random consistency (RI)

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.0 | 0.0 | 0.58 | 0.90 | 1.12 | 1.24 | 1.32 | 1.41 | 1.45 | 1.49 |

If $C R \leq 0.1$, the judgement matrix is acceptable otherwise it is considered inconsistent, and the entries that are given by the decision maker have to be revised until a satisfactory consistency ratio is obtained.

### 2.5.5 Synthetic utility

Once the alternatives are compared with each other in terms of each one of the decision criteria and the individual priority vectors are derived, the synthesis step is taken.

The priority vectors become the columns of the decision matrix. The weights of importance of the criteria are also determined by using pair-wise comparisons. Therefore, if a problem has $m$ alternatives and $n$ criteria, then the decision maker is required to construct $n$ judgment matrices (one for each criterion) of order $m \times m$ and one judgment matrix of order $n \times n$ (for the $n$ criteria).

Finally, given a decision matrix the final priorities, denoted by $A_{i}^{A H P}$, of the alternatives in terms of all the criteria combined are determined according to the following formula:

$$
\begin{equation*}
A_{i}^{A H P}=\sum_{j=1}^{m} v_{i j} \cdot w_{j}, \text { for } i=1, \ldots, n \tag{2.5}
\end{equation*}
$$

where $v_{i j}$ describes the performance of alternative $a_{i}$ against criterion $c_{j}$, and $w_{j}$ indicates the relative importance of one criterion $c_{j}$.

The global priorities thus obtained are used for final ranking of the alternatives and selection of the best one.

## Example 2.1

Let us treat a problem of purchasing a car. Suppose that this problem involves three criteria, and four alternatives as shown in Figure 2.1.

To apply the AHP method, we must follow these main steps :

## 1. Pair-wise comparison:

The first step in AHP is to calculate the relative importance of the different criteria. We provide an initial matrix (see Table 2.4) for the pair-wise comparison criteria in which the principal diagonal contains entries of 1, as each factor is as important as itself. For instance, when fuel economy criterion is compared to style criterion then the decision maker has determined that fuel economy is between to be classified "somewhat more important" than style. Thus, the corresponding


Figure 2.1: Hierarchy of car choice AHP model
comparison assumes the value of 3. A similar interpretation is true for the rest of the entries.

Table 2.4: Weights on criteria

| Criteria | Style | Reliability | Fuel Economy |
| :---: | :---: | :---: | :---: |
| Style | 1 | $\frac{1}{2}$ | 3 |
| Reliability | 2 | 1 | 4 |
| Fuel Economy | $\frac{1}{3}$ | $\frac{1}{4}$ | 1 |

Our next step is to evaluate all the alternative on each criterion. For instance, if we take style criterion, then we might get the following matrix (see Table 2.5):

Table 2.5: Comparison matrix for style criterion

| Style | Mazda | Ford | Renault | Citroen |
| :---: | :---: | :---: | :---: | :---: |
| Mazda | 1 | $\frac{1}{4}$ | 4 | $\frac{1}{6}$ |
| Ford | 4 | 1 | 4 | $\frac{1}{4}$ |
| Renault | $\frac{1}{4}$ | $\frac{1}{4}$ | 1 | $\frac{1}{5}$ |
| Citroen | 6 | 4 | 5 | 1 |

## 2. Priority vector:

For each pair-wise comparison matrix, we use the eigen vector method to get the priority vector. For the criteria matrix, for example, we get the following priority vector (see Table 2.6):

Table 2.6: Computing the criteria priority values

| Criteria | Style | Reliability | Fuel Economy | Priority |
| :---: | :---: | :---: | :---: | :---: |
| Style | 1 | 0.5 | 3 | 0.3196 |
| Reliability | 2 | 1 | 4 | 0.5584 |
| Fuel Economy | 0.33 | 0.25 | 1 | 0.1210 |

## 3. Consistency ratio:

Perfect consistency rarely occurs in practice. Ratings should be consistent in two ways: First, Ratings should be transitive. That means that if "Reliability" is better than "Style", and "Style" is better than "Fuel Economy", then "Reliability" must be better than "Fuel Economy". Second, ratings should be numerically consistent. For example, we know that "Reliability $=3$ Style" and "Reliability $=5$ Fuel Economy" that means that "Style $=(5 / 3)$ Fuel Economy".
To calculate the consistency ratio we must solve:

$$
A w=\lambda_{\max } \cdot w
$$

by solving

$$
\operatorname{det}(\lambda I-A)=0
$$

We get $\lambda_{\max }=3.0180$. Then, the CI index is found by:

$$
C I=\left(\lambda_{\max }-n\right) /(n-1)=0.009
$$

The final step is to calculate the $C R$ by using the table derived from Saaty's book (Table 2.3).

$$
C R=C I / R I=0.0090 / 0.58=0.01552
$$

where $R I=0.58$ because the pair-wise comparison matrix is a matrix of order 3.
$C R$ value is less than 0.1, so the evaluations are consistent. A similar procedure is repeated for the rest of matrix.

Table 2.7: Decision matrix

| Criteria | Style | Reliability | Fuel Economy | Utility |
| :---: | :---: | :---: | :---: | :---: |
|  | 0.3196 | 0.5584 | 0.120 | 1 |
| Ford | 0.1160 | 0.3790 | 0.3010 | 0.2854 |
| Mazda | 0.2470 | 0.2900 | 0.2390 | 0.2700 |
| Renault | 0.2600 | 0.740 | 0.2120 | 0.0864 |
| Citroen | 0.5770 | 0.2570 | 0.2480 | 0.3582 |

## 4. Synthetic utility:

The next step is to calculate the global priorities to obtain the final ranking of alternatives and to select the best one. So, to determine these final scores, we will apply the Equation (2.5) by multiplying the criteria weights' by the ratings for the decision alternatives for each criterion, and summing the respective products (see Table 2.7).

As a result, the citroen car will be preferred since it has the highest values.

### 2.6 AHP method under uncertainty

Standard versions of the AHP method gives good results in a context in which everything is known with certainty. However, the reality is connected to uncertainty and imprecision by nature. A decision maker may encounter several difficulties when expressing his own level of preferences between alternatives or also criteria. These difficulties arise due to different situations. Such uncertainty may badly affect the final decision making.

Moreover, in such uncertain and imprecise context, standard AHP cannot be applied. So, it is inadequate and badly adapted to ensure its role.

In order to overcome these difficulties and to extend the AHP on a more real elicitation procedure, several AHP methods are combined within uncertain theories such as probability theory, fuzzy set theory, belief function theory and possibility theory.

One of these extensions is the Fuzzy AHP appeared in (Laarhoven \& Pedrycz, 1983), which utilized triangular fuzzy numbers to model the pairwise comparisons. Since then, several fuzzy AHP developments have been proposed (Lootsma, 1997). Besides, probabilistic AHP methods are intro-
duced in (Basak, 1998), handling pair-wise comparisons matrices based on probability theory, where each element of which is the prior probability.

In particular in the belief function framework, (Beynon et al., 2000) have proposed a method called the DS/AHP method comparing not only single alternatives but also groups of alternatives. Besides, Smarandache et al. (Dezert et al., 2010) have developed the DSmT/AHP which is based on the Dezert-Smarandache theory (Smarandache \& Dezert, 2004). This method aimed at performing a similar purpose as DS/AHP that is to compare groups of alternatives.

### 2.6.1 Fuzzy AHP method

AHP method has been criticized because it cannot handle the inherent uncertainty and imprecision which are associated to mapping of decision maker perceptions to exact numbers. So, AHP requires exact or crisp numbers. That's why, many fuzzy AHP methods are proposed by various authors (Laarhoven \& Pedrycz, 1983) (Lootsma, 1997).

Fuzzy AHP approach use triangular fuzzy numbers to model the pairwise comparisons (Laarhoven \& Pedrycz, 1983).

A fuzzy number is a special fuzzy set $F=\{(x, \mu(x)), x \in \Re\}$, where $x$ takes its values on the real line, and $\mu(x)$ is a continuous mapping from $\Re$ to the closed interval $[0,1]$.

A triangular fuzzy number is denoted by $\tilde{M}=(a, b, c)$, where $a \leq b \leq c$, has the following triangular-type membership function:

$$
\mu_{\tilde{M}}(x)= \begin{cases}0, & x<a  \tag{2.6}\\ \frac{x-a}{b-a} & , a \leq x \leq b \\ \frac{c-x}{c-b}, & b \leq x \leq c \\ 0, & x>c\end{cases}
$$

As a result, the ratio comparison between the relative preference of elements of the hierarchy can be modelled through a fuzzy scale value associated with a degree of fuzziness as shown in Table 2.8.

By using triangular fuzzy numbers, via pair-wise comparison, a fuzzy judgment matrix is then constructed. As a solution and in oder to compute

Table 2.8: The fuzzy Saaty scale

| Intensity of importance | Definition |
| :---: | :---: |
| $\tilde{1}=(1,1,3)$ | Equal importance |
| $\tilde{3}=(1,3,5)$ | Somewhat more important |
| $\tilde{5}=(3,5,7)$ | Much more important |
| $\tilde{7}=(5,7,9)$ | Very much more important |
| $\tilde{9}=(7,9,9)$ | Absolutely more important |
| $\tilde{2}, \tilde{4}, \tilde{6}, \tilde{8}$ | Intermediate values |

the priority vector, the fuzzy eigen value is then used.

### 2.6.2 DS/AHP method

Though the popularity and efficiency of the AHP approach, this method is often criticized, because it cannot be applied in an uncertain and imprecise context. In fact, in some cases, the decision maker cannot make pair-wise comparisons between all the alternatives.

To solve this problem, (Beynon et al., 2000) (Beynon, 2002) propose to extend the AHP on a more real elicitation procedure. Beynon et al. have proposed a method called the DS/AHP method, which extended the AHP approach with belief function theory to compare not only a single alternatives but also groups of alternatives between each other.

Within DS/AHP method, for each criterion, there are certain groups of decision alternatives, including $\Theta$, about which the decision maker can express some degree of favourable knowledge.

Through comparing a group of alternative to $\Theta$, the decision maker will express some degree of favourable knowledge on each of these groups of alternative. This differs from the AHP method that makes pair-wise comparisons between individual decision alternatives (Beynon et al., 2000) (Beynon et al., 2001), here each group of alternatives identified is compared to all possible alternatives in the frame of discernment.

After identifying the candidate sets of criteria, what is left is setting priorities of the sets of alternatives. At this point, classical pair-wise comparisons of the elements are made to obtain these priorities.

In DS/AHP method, the Saaty's scale was modified for simplicity, it is reduced to 5 unit scale (see Table 2.9).

Table 2.9: Knowledge/favourable scale

| Opinion/Knowledgeable | Numerical rating |
| :---: | :---: |
| Extremely favourable | 6 |
| Strongly to extremely | 5 |
| Strongly favourable | 4 |
| Moderately to strongly | 3 |
| Moderately favourable | 2 |

To calculate the priority vector, the weight of criteria must be incorporated in the pair-wise comparison matrix. This is done by multiplying the elements in the last column (except the last entry in that column) by the respective importance value for that criterion. If $p$ is the weight of the criterion $j$ and $v_{i j}$ is the favourability opinion for a particular group of alternatives, then the resultant value is $p \times v_{i j}$ (the resultant change in the bottom row of the matrix is similarly $\left(1 /\left(p \times v_{i j}\right)\right)$.

After computing the priority vector, the priority values in each column sum to one. As a result, Beyon et al. consider that these priorities values are a basic belief assignment.

Then, the obtained priorities vectors are combined using the Dempster's rule of combination to integrat them into a single bba. In fact, Beyon et al. assume that criteria are independent pieces of evidence, offering information on the decision makers knowledge towards the favourability of the identified groups of alternatives, hence the associated bba's are independent.

Finally, to choose the best alternative, the presented approach uses the belief and plausibly functions.

## Example 2.2

To describe the DS/AHP method, we will use a simple example. Indeed, the decision involves buying a new car, from a choice of three well-known types of car ( $A, B$ and $C$ ). However, there are four criteria: price, fuel economy, comfort and style that are believed influence the choice of car. Hence, the overall objective is to decide which is the best car to buy.

First, we establish the hierarchy frame for purchasing a car, as shown in Figure 2.2. So, for each criterion, there are certain groups of decision alternatives, including $\Theta$.


Figure 2.2: Hierarchy of modified car choice model

To calculate the weigth of criteria, the standard AHP method is applied to get the priority vector (see Table 2.10).

Table 2.10: Criteria priority values

| Criterion | Price | Fuel | Comfort | Style |
| :---: | :---: | :---: | :---: | :---: |
| Priority | 0.3982 | 0.0851 | 0.2159 | 0.2988 |

Also, the same process is used to get the alternatives priorities. For example, for the comfort criterion we get (Table 2.11):

Table 2.11: Initial pair-wise matrix for comfort criterion

| Comfort | $\{A\}$ | $\{B, C\}$ | $\{A, B, C\}$ |
| :---: | :---: | :---: | :---: |
| $\{A\}$ | 1 | 0 | 4 |
| $\{B, C\}$ | 0 | 1 | 6 |
| $\{A, B, C\}$ | $1 / 4$ | $1 / 6$ | 1 |

The zero's which appears in the knowledge matrix indicates no attempt to assert knowledge between groups of decision alternatives.

Next, to calculate the priority vector, we multiply each element of the pair-wise matrices (except the last entry in that column) by the respective importance value for that criterion.

Let us continue with the comfort criterion, which had an importance value $p=0.2159$, we obtain Table 2.12:

Table 2.12: Pair-wise matrix for comfort criterion after influence of its priority rating

| Comfort | $\{A\}$ | $\{B, C\}$ | $\{A, B, C\}$ |
| :---: | :---: | :---: | :---: |
| $\{A\}$ | 1 | 0 | 0.8714 |
| $\{B, C\}$ | 0 | 1 | 1.3072 |
| $\{A, B, C\}$ | 1.1475 | 0.7650 | 1 |

Table 2.13: Pair-wise matrix for price, fuel and style

| Price | $\{A, B\}$ | $\Theta$ |  |
| :---: | :---: | :---: | :---: |
| $\{A, B\}$ | 1 | 6 |  |
| $\Theta$ | $1 / 6$ | 1 |  |
| Fuel | $\{B\}$ | $\Theta$ |  |
| $\{B\}$ | 1 | 3 |  |
| $\Theta$ | $1 / 3$ | 1 |  |
| Style | $\{A\}$ | $\{C\}$ | $\Theta$ |
| $\{A\}$ | 1 | 0 | 5 |
| $\{C\}$ | 0 | 1 | 2 |
| $\Theta$ | $1 / 5$ | $1 / 2$ | 1 |

For the other three criteria, we get Table 2.13:
In these knowledge matrices the right eigenvector method is again used to calculate the priority values (see Table 2.14). That is, the normalised elements of the eigenvector associated with the largest eigenvalue from the matrix.

Table 2.14: Priority values

| Price | Priority | Fuel | Priority | Comfort | Priority | Style | Priority |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\{A, B\}$ | 0.7054 | $\{B\}$ | 0.2037 | $\{A\}$ | 0.2419 | $\{A\}$ | 0.4265 |
| $\Theta$ | 0.2946 | $\Theta$ | 0.7963 | $\{B, C\}$ | 0.3628 | $\{C\}$ | 0.1706 |
|  |  |  |  | $\Theta$ | 0.3953 | $\Theta$ | 0.4029 |

The priority values in each column sum to one. These are directly defined as basic belief assignments. So, for comfort criterion, we note $m_{c}$ as the associated bba:

$$
m_{c}(\{A\})=0.2419, m_{c}(\{B\})=0.3628, m_{c}(\Theta)=0.3953
$$

We can go through a similar process with price, fuel and style (see Table 2.15). These bba's are independent pieces of evidence. Hence, the associated bba are independent. So, we use the Dempster's rule of combination to get a single bba and the resulting bba is then:

Table 2.15: The bba $m_{\text {car }}$ after combining all evidence

| DA | $\{A\}$ | $\{B\}$ | $\{C\}$ | $\{A, B\}$ | $\{B, C\}$ | $\Theta$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $m_{\text {car }}$ | 0.4668 | 0.2292 | 0.0482 | 0.1420 | 0.0545 | 0.0593 |

Now, we can calculate the belief and plausibility functions:

Table 2.16: Belief and plausibility values for subsets of cars

| Cars | Bel | Pl |
| :---: | :---: | :---: |
| $\{A\}$ | 0.4309 | 0.6650 |
| $\{B\}$ | 0.2312 | 0.5180 |
| $\{C\}$ | 0.0511 | 0.1729 |
| $\{A, B\}$ | 0.8271 | 0.9489 |
| $\{A, C\}$ | 0.4820 | 0.7688 |
| $\{B, C\}$ | 0.3350 | 0.5691 |

If we consider the focal element $\{A\}$, then there is a small amount of evidence in favour as well as against the hypothesis $A$ being the best car. For $\{C\}$ the table shows strong evidence against being the best choice. Interestingly the set $\{A, B\}$ shows strong evidence in favour of including within its elements the best choice of car.

### 2.7 Conclusion

In this chapter, we have presented the basic concepts of the AHP method. We have defined its procedure. Then, we have given an example to explain this approach.

Despite the advantages of AHP method, several researches are focusing on improving more and more the results provided by this approach, especially, in an environment where uncertainty may exist in the different levels relative to a decision making problem. One of these extensions is the fuzzy AHP
approach and also the DS/AHP approach, and many researches are still needed in order to deal with the uncertainty.

Thus, our objective will be the adaptation of this method to the belief function theory and to develop what we call belief AHP approach that will be detailed in the next chapter.

## Part II

## Belief AHP Approach

## Chapter 3

## Belief AHP approach

### 3.1 Introduction

As mentioned in Chapter 2, the Analytic Hierarchy Process (AHP) is considered as one of the most well known Multi-Criteria Decision Making (MCDM) method, due to its ability to solve complex problems by breaking them into smaller and smaller parts. The strength of AHP is that it is easier to understand and it can effectively handle both qualitative and quantitative data.

However, standard AHP is criticized because, in an environment characterized by uncertain and imprecise data this method does not perform well. In fact, the decision maker may express his preferences with some degrees of uncertainty, that means he may have a certain doubt or a lack of knowledge about alternatives or criteria.

To overcome this limitation, we propose, as a solution, to develop what we call a belief AHP method, a new MCDM method based on the AHP approach in order to cope with uncertain preferences.

Firstly, we have developed a method that deals with uncertainty into two levels namely the criterion and alternative levels. So, the decision maker may express his preferences with some degrees of uncertainty.

Secondly, we have extended our proposed approach on a more flexible method that integrates additional uncertain and/or imprecise knowledge regarding the comparison procedure.

Our methods are based on the belief function theory as understood in the Transferable Belief Model (TBM). This theory provides a convenient framework for dealing with incomplete and uncertain information, notably those given by experts.

In this chapter, we give an insight into our proposed methods. We first give some motivations to develop the belief AHP method for handling uncertainty. Next, we define the objectives of the belief AHP approach and we explain the uncertainty concealed in this standard method. Then, we detail each of the proposed methods and we give some examples to illustrate them.

### 3.2 Definition and motivations

A belief AHP method is an extension of the AHP approach in an uncertain environment. The uncertainty will be represented and handled by the means of the belief function theory as explained in the TBM.

Contrary to the standard AHP approach where the preferences of the decision maker are known with certainty, in a belief AHP method these preferences may be uncertain and imprecise. Such uncertainty can appear either in the criterion or in the alternative levels. In a first part, our method will be able to compare groups of alternatives instead of comparing only single alternatives between each other. In a second part, to judge the importance of criteria, our approach offers a formalism allowing the expert to express his ranking even over subgroups of criteria. Thus, our approach deals with uncertainty into two levels namely the criterion and alternative levels.

### 3.3 Uncertainty in the AHP method

Though its popularity and efficiency, the classical AHP is criticized (Joaquin, 1990) (Holder, 1995) because, in real-life decision making situation, the decision maker may encounter several difficulties when expressing his own level of preferences between alternatives or also criteria.

These difficulties arise due to different situations. The first cause is due to incomplete or lack of data for making decisions. In fact, the decision maker cannot ensure pair-wise comparison between all the decision alternatives because sometimes the information about them may be incomplete due to the
time pressure and the lack of data.
Second, the elicitation of preferences may be rather difficult when the number of alternatives is large. If the number of alternatives in the hierarchy increases then, more comparisons are needed to be made.
In addition, in some practical problems, experts are able to compare only subsets of alternatives and cannot compare separate or singleton alternatives. Suppose that there is a lot of transport facilities which can be divided into three groups: motor transport, air transport and water transport. An expert cannot provide pair-wise comparisons of all the facilities, but he can say that motor transport is more preferable than water transport (Utkin \& Simanova, 2008).

Furthermore, an expert should also make a ranking of the importance of criteria. In many situations, it is easier for him to express his opinions and comparisons between subsets of criteria and not compulsory on singleton criterion. For example, for the fuel, we may get two subgroups: the first one containing both natural gas and gasoline criteria and the second one only singleton criterion which is diesel.

As a result, in such uncertain and imprecise context, standard AHP cannot be applied.

### 3.4 Objectives

The objective of this work is to develop a new concept that we will call belief AHP method. In addition to the objectives of the standard AHP method, the belief AHP aims at realizing two major objectives:

- Building an order of preference on a number of decision alternatives, that is to obtain priorities from uncertain and imprecise preferences' sets on all the levels of the hierarchy in AHP decision problem (criteria and alternatives).
- choosing the best alternative: the uncertainty will be taken into account in the final decision.

This new approach is based on both the AHP method and the belief function theory in order to cope with uncertainty.

### 3.5 Estimation of pertinent data

### 3.5.1 Data acquisition

One of the most crucial steps in many decision making methods is the accurate estimation of the pertinent data. This is a problem not only found in the AHP method, but it is crucial in many other methods which need to elicit preferences from the decision maker. Very often preferences cannot be known in terms of absolute values.

Generally, any MCDM method is constructed from two levels of judgment. The first one is used to represent the weight of criteria and the second one is composed of alternatives where for each alternative, we know exactly its performance against each criterion.

However, due to the uncertainty, the structure of these data may be different from the traditional one. In fact, AHP method elicits preferences from decision maker's answers. As mentioned above, these preferences may be incomplete and imprecise because, in real-life decision making situation, the decision maker may have a certain doubt or a lack of knowledge about alternatives and also criteria. As a result, he may encounter several difficulties when expressing his own level of preferences between these alternatives and criteria.

As a solution, we propose to represent this uncertainty by comparing groups of alternatives instead of comparing only single alternatives between each other. Then, we assume that the obtained priorities are transformed into basic belief assignments (bba's). Each bba represents the opinions-beliefs of the decision maker about his preferences. On the other hand and in order to judge the importance of criteria, our method offers a formalism allowing the expert to express his ranking even over subgroups of criteria.

Among the advantages of working under the belief function framework, is that this theory provides a convenient framework for dealing with individual elements of the hypothesis set as well as their subsets.

## Example 3.1

This is an example, to illustrate our new structure of the data: data acquisition.

We will use an everyday situation related to the fact of buying a car. We assume there are three alternatives $A, B$ and $C$ that will be compared according to four criteria: price $(P)$, fuel ( $F$ ), comfort ( $C$ ) and style ( $S$ ). Hence, our major objective is to make up mind about the suitable car to be bought.

Table 3.1: Data acquisition

|  | Price | Fuel | Comfort | Style |
| :---: | :---: | :---: | :---: | :---: |
| $A$ | 25000 | $?$ | Average | economic |
| $B$ | 30000 | 7.5 | $?$ | luxe |
| $C$ | $?$ | $?$ | $?$ | $?$ |

In Table 3.1, the judgments made by the decision maker are given over the four criteria: $(P),(F),(C)$ and $(S)$. For a single criterion, values in the column related to the performance of an alternative against the specific criterion, where no value is given, it may be due to the lack of knowledge about alternatives.

To illustrate our approach, for the criterion ( $P$ ) for example, alternatives which have the same performance value are grouped together. In other terms, we will get two subsets of alternatives: $\{A, B\}$ and $\{A, B, C\}: A$ and $B$ have almost the same degree of preferences and the set $\{A, B, C\}$ to express the ignorance.

The next step is to assign an important level to each criterion. The decision maker assigns a degree of preferences for each criterion. Criteria which have the same degree of preference are grouped together. We will get the following sets of criteria: $\{P\},\{F\}$ and $\{C, S\}$. The (C) and (S) criteria are grouped together because they have the same degree of preference. Figure 3.1 resumes the candidate alternatives and criteria.

### 3.5.2 Special cases

Within the belief function framework, two extreme cases such as the total ignorance and the total knowledge can be easily expressed.
For the criterion level:

1. When the preferences of the decision maker are perfectly known and


Figure 3.1: The hierarchy frame
are unique. This case will be represented by a single criteria instead of group of criteria.
2. When the expert is not able to give any information about criteria. So, we will get a single set of criteria which all of them have the same degree of preferences. This case is referred to as total ignorance.

For the alternative level:

1. When the preferences of the decision maker are perfectly known, and he is able to compare single alternatives between each other instead of sets of alternatives. This case will be represented by a singleton alternative.
2. When the expert is not able to give any information about alternatives. Thus, the obtained bba will be a vacuous basic belief assignment (see Section 1.3.1), and we will get a single subset $\Theta$. This case is referred to as total ignorance.

### 3.6 Belief AHP approach

### 3.6.1 Introduction

As with standard AHP method, building a belief AHP falls to the definition of its fundamental steps seen in the previous chapter, namely, hierarchical model, pair-wise comparisons, local priorities and global priorities. These concepts must take into account the uncertainty encountered in the obtained data.

### 3.6.2 Computational procedure of the belief AHP approach

Since impression and uncertainty are common characteristics in many decision making problems, a belief AHP method should be able to deal with this uncertainty. In other words, the conventional AHP allows the decision maker to express his preference on a number of decision alternatives, that is to obtain priorities of the preferences' sets on all the levels of the hierarchy in AHP decision problem (criteria and alternatives).

Due to the uncertainty and contrary to the traditional judgment procedure where it includes only certain preferences, the structure of our obtained data may be different from the traditional one. So, the decision maker may have a certain doubt or a lack of knowledge about alternatives and also criteria. Indeed, in many practical problems, the uncertainty may appear into two levels namely the criteria and the alternatives. Thus to solve this problem, we propose a method for solving complex problems under the condition that it tolerates imprecision and uncertainty when the expert expresses his preferences between criteria and also alternatives. In other words, our approach will be able to compare groups of criteria and also groups of alternatives between each other.

## Identification of the candidate criteria

One of the key questions being issued over the implementation of any MCDM problem is the identification of the candidate criteria.

By nature, the importance of criteria is relative to each other. There-
fore, a decision maker may encounter some difficulties to compare separate ones. Thus, he can give subjective and imprecise assessments. In this case, conventional AHP seems inadequate to determine the relative weight.

In our work, and in order to overcome this difficulty, a new method for judging the importance of these criteria is proposed, instead of the standard one. In fact, we suggest to extend the AHP method to an imprecise representation rather than forcing the decision maker to provide precise representations of imprecise perceptions.

We suppose that there is a set of criteria $\Omega=\left\{c_{1}, \ldots, c_{m}\right\}$ consisting of $m$ elements. Denote the set of all subsets of C (the power set) by $2^{\Omega}$, and let $C_{k}$ be the short notation of a subset of $C$, i.e., $C_{k} \subseteq C$ and $C_{k} \in 2^{\Omega}$.

An expert chooses a subset $C_{k} \subseteq C$ of criteria from the set C and compares this subset with another subset $C_{j} \subseteq C$. In other terms, the decision maker expresses his preferences by comparing these subsets of criteria $C_{k} \succ C_{j}$, which means that an expert chooses $C_{k}$ from $C_{j}$, and $C_{k}$ is more preferable than $C_{j}$. Thus, criteria that belong to the same group have the same degree of preferences.

Since we are not performing pair-wise comparisons of criterion but relating groups of criteria, these sets of criteria should not consider a criterion in common, because if one criterion is included in two groups, then each group will give a different level of favorability.

The decision maker compares these subsets of criteria, and he provides preference values according to the Saaty's scale.

By generalization, the subsets of criteria can be defined as:

$$
\begin{equation*}
C_{k} \succ C_{j}, \forall k, j \mid C_{k}, C_{j} \in 2^{\Omega}, C_{k} \cap C_{j}=\emptyset \tag{3.1}
\end{equation*}
$$

To conclude, the main idea is to allow the expert to express his opinions on groups of criteria instead of single one. It means that our method is based on a measure of preferences between criteria. So, an expert chooses a subset of criteria by assuming that criteria having the same degree of preference are grouped together. If an expert chooses a group of criteria, then we could suppose that all of them have the same importance and consequently have the same distributed weights.

## Identification of the candidate alternatives

In many complex problems, experts are able to compare only subsets of alternatives and cannot evaluate separate alternatives in these subsets, or in the same case, expert may have an incomplete decision matrix.

To solve this problem, that means to reduce the number of alternatives which decrease the number of comparisons, our method suggests not to consider all of them but just to choose groups of those alternatives. In other terms, the decision maker compares not only a single alternative but also sets of alternatives between each other.

One of the possible solutions of this task is to use the DS/AHP method (Beynon et al., 2000) (Beynon, 2002). According to this idea, the decision maker is able to compare groups of alternatives instead of single one. He has to identify favorable alternatives from all the set of the possible ones. As explained in the previous chapter, this method allows measures of uncertainty and ignorance to be calculated on the judgment made by the decision maker.

Similarly to the criterion level, we assume that there is a set of alternatives $\Theta=\left\{a_{1}, \ldots, a_{n}\right\}$ consisting of $n$ elements. Denote the set of all subsets of $A$ (the power set) by $2^{\Theta}$, and let $A_{k}$ be the short notation of a subset of A, i.e., $A_{k} \subseteq A$ and $A_{k} \in 2^{\Theta}$.

The decision maker expresses his preferences by comparing subsets of alternatives including $\Theta$, about which the expert can express his preferences. For example, $A_{k} \succ A_{j}$, which means that $A_{k}$ is more preferable than $A_{j} . \Theta$ as a set of alternatives allows the decision maker to express his ignorance. The main aim behind this method was explained in the previous chapter (Beynon et al., 2000) (Beynon, 2002).

## Computing the weight of considered criteria

After identifying the candidate sets of criteria, what is left is "how to calculate the weight of these criteria?". At this point, classical pair-wise comparisons of the elements are made to obtain these priorities.

In this study, we have adopted the Saaty's scale (see Table 2.2) to evaluate the importance of pairs of grouped elements in terms of their contribution. Thus, the comparison matrix is modified to take into account the partial uncertainty by comparing both singletons and disjunctions elements. The
priority vectors are then generated using the eigenvector method.
The objective is then to find the eigen vector $w$ for each pair-wise comparison matrix. According to Saaty, the eigen vector can be generated in different ways, and amongst the most used one is the geometric means (Saaty, 2003).

If $A=\left[a_{i j}\right]$ is a pair-wise comparison matrix, then the geometric means for each row is defined by:

$$
\begin{equation*}
r_{i}=\prod_{j=1}^{l} a_{i j}^{1 / l} \tag{3.2}
\end{equation*}
$$

and the weight $w_{i}$ as the components of the normalized eignvector, where:

$$
\begin{equation*}
w_{i}=\frac{r_{i}}{r_{1}+r_{2}+\ldots+r_{l}} \tag{3.3}
\end{equation*}
$$

where $l$ the number of sets of criteria.

## Example 3.2

If we consider the previous example where: $\{P\},\{F\}$ and $\{C, S\}$ are the groups of criteria which belong to the power set of $2^{\Omega}$. The decision maker can define his preferences by using the pair-wise comparison of the following type "How important is criterion $\{P\}$ relative to criterion $\{F\}$ ?". To respond to this question, the decision maker uses the following Saaty's scale expressing the intensity of the preference for one criterion versus another.

Table 3.2: Pair-wise comparison matrix for the criterion level

| Criteria | $\{P\}$ | $\{F\}$ | $\{C, S\}$ | Priority $(w)$ | Nomalized vector |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\{P\}$ | 1 | 2 | 6 | 0.58 | 1 |
| $\{F\}$ | $\frac{1}{2}$ | 1 | 4 | 0.32 | 0.55 |
| $\{C, S\}$ | $\frac{1}{6}$ | $\frac{1}{4}$ | 1 | 0.1 | 0.17 |

To get the priority vector, we apply the geometric means formula:

1. Multiply out each row of the matrix.
2. Since there are l entries (the number of sets of criteria) in each row, take the $l^{\text {th }}$ root of multiplication.
3. Normalize these roots by deriving the total and dividing them by the total.

Let us show how to compute the eigen vector:

$$
\begin{aligned}
& x=\sqrt[3]{1 \times 2 \times 6}=2.29 \\
& y=\sqrt[3]{\frac{1}{2} \times 1 \times 4}=1.26 \\
& z=\sqrt[3]{\frac{1}{6} \times \frac{1}{4} \times 1}=0.34 \\
& \text { total }=x+y+z \\
& \text { then, } w=\left(\begin{array}{c}
\frac{x}{\text { total }} \\
\frac{y}{\text { total }} \\
\text { total }
\end{array}\right)=\left(\begin{array}{c}
0.58 \\
0.32 \\
0.1
\end{array}\right)
\end{aligned}
$$

From Table 3.2, we conduct the preferred order of proposed criteria (the importance) based on their final priorities values as follows: $\{P\} \succ\{F\} \succ$ $\{C, S\}$, where $\{P\}$ is preferred to $\{F\}$ and $\{F\}$ is preferred to $\{C, S\}$.

Then, we propose to calculate the normalized vector, which is given by the division of the importance of criteria by the maximum of priorities. For each $w_{i}$, we get:

$$
\text { normalized } w_{i}=\frac{w_{i}}{w_{\max }}
$$

## Computing the alternatives priorities'

In standard version of AHP method, all the alternatives are evaluated regarding each criterion. The comparison takes this form: "how important is alternative $A_{k}$ when compared to alternative $A_{j}$ with respect to criterion $c_{1}$ ?"

As in the DS/AHP method (Beynon et al., 2000) (Beynon, 2002), uncertain comparison matrices are constructed with respect to each criterion. That means, classical pair-wise matrix is used to compare sets of alternatives between each other with respect to a specific criterion instead of comparing only single alternative. So, the same procedure used in the previous step is adopted to calculate the priority of the considered alternatives.

## Example 3.3

Let us continue with Example 3.1. A sample matrix for the price criterion, for example, being shown in Table 3.3.

Table 3.3: Comparison matrix for price criterion

| P | $\{A, B\}$ | $\{A, B, C\}$ | Priority |
| :---: | :---: | :---: | :---: |
| $\{A, B\}$ | 1 | 9 | 0.896 |
| $\{A, B, C\}$ | $\frac{1}{9}$ | 1 | 0.104 |

## Updating the alternatives priorities'

Once the priorities of decision alternatives and criteria are computed, we have to define a rule for combining them. The problem here is that we have priorities concerning criteria and groups of criteria instead of single ones, whereas the sets of decision alternatives are generally compared pairwise with respect to a specific single criterion. Within this structure of alternatives and criteria, our belief AHP method cannot use the strategy used by the standard method which aggregates all local priorities from the decision table by a simple weighted sum.

In order to overcome this difficulty, we choose to apply the belief function theory because it provides a convenient framework for dealing with individual elements of the hypothesis set as well as their subsets.

At the decision alternative level, we propose to represent the uncertainty on the decision maker preferences over the set of alternatives by a bba defined on the set of possible alternatives. In fact, within our framework, we have $A_{k} \subseteq 2^{\Theta}$ and we have the priority values of each $A_{k}$ in each comparison matrix representing the opinions-beliefs of the expert about his preferences. So, we assume that the set of alternatives is the frame of discernment, and we notice that the priority vector sums to one which can be considered as a bba $\left(m\left(A_{k}\right)\right)$ which represents its power set.

Given a pair-wise comparison matrix which compares the sets of alternatives according to a specific criterion. For each set of alternatives $A_{k} \in 2^{\Theta}$ and $A_{k}$ belongs to this pair-wise matrix, we get:

$$
\begin{equation*}
m\left(A_{k}\right)=w_{k} \tag{3.4}
\end{equation*}
$$

where $w_{k}$ is the eigen value of the $k^{\text {th }}$ sets of alternatives.
The next step is to combine the obtained bba with the importance of their respective criteria to measure their contribution. In this context, our approach proposes to regard each priority value of a specific set of criteria as a measure of reliability. In fact, this factor is used to update experts' beliefs (bba) by taking into account the important of each set of criteria. The idea is then to measure most heavily the bba evaluated according to the most important criteria and conversely for the less important ones.

If we have $C_{k}$ (as defined above) a subset of criteria, then we get $\beta_{k}$ its corresponding measure of reliability.

$$
\begin{equation*}
\beta_{k}=\text { normalized } w_{k}=\frac{w_{k}}{w_{\max }} \tag{3.5}
\end{equation*}
$$

As a result, two cases will be presented: First, if the reliability factor represents a single criterion, then the corresponding bba will be directly discounted.

If $C_{k}$ is a singleton criterion, then we apply the discounting rule and we get:

$$
\begin{align*}
m_{C_{k}}^{\alpha_{k}}\left(A_{j}\right) & =\beta_{k} \cdot m_{C_{k}}\left(A_{j}\right), \forall A_{j} \subset \Theta  \tag{3.6}\\
m_{C_{k}}^{\alpha_{k}}(\Theta) & =\left(1-\beta_{k}\right)+\beta_{k} \cdot m_{C_{k}}(\Theta) \tag{3.7}
\end{align*}
$$

where $A_{j}$ a subset of alternatives that are evaluated with respect to the criterion $C_{k}, m_{C_{k}}\left(A_{k}\right)$ the relative bba for the subset $A_{k}, \beta_{k}$ its corresponding measure of reliability, and we denote $\alpha_{k}=1-\beta_{k}$.

Second, if this factor represents a group of criteria, their corresponding bba's must be combined using the conjunctive rule, then it will be discounted by the measure of reliability relative to this group of criteria.

In other terms, we consider that each group of criteria has a set of pairwise comparison matrix. That means, each element of a specific group of criteria has its own pair-wise matrix that evaluates the sets of alternatives with respect to this specific criterion. As mentioned above, our main purpose is to compare the sets of decision alternatives regarding certain groups of criteria. Therefore, our proposed approach assumes that each pair-wise comparison matrix is considered as a distinct source of evidence, which provides information on opinions towards the preferences of particular decision alternatives. Then, based on the belief function framework, we can apply the conjunctive rule of combination to obtain a single representation value
of these different bba's. As a result, the obtained bba compares the sets of alternatives according to this set of criteria.

Let $C_{k}$ a subset of criteria, and $c_{i} \in C_{k}$, then we apply the conjunctive rule of combination to obtain $m_{C_{k}}$ :

$$
\begin{equation*}
m_{C_{k}}=® m_{c_{i}}, \quad i=\{1, \ldots, h\} \tag{3.8}
\end{equation*}
$$

where $h$ is the number of element of a specific group of criteria $C_{k}$ and $c_{i}$ is a singleton criterion.

Finally, these obtained bba's ( $m_{C_{k}}$ ) will be discounted by their corresponding measure of reliability $\beta_{k}$. We apply the same idea used in Equation 3.6 and 3.7, to get $m_{C_{k}}^{\alpha_{k}}$.

## Example 3.4

Let us continue with the previous examples. From Table 3.3, we can notice that the priority value is sum to one. So, we suppose that these priorities are the bba's. We denote these bba's by $m_{p}$ and we get:

$$
m_{p}(\{A, B\})=0.896 \text { and } m_{p}(\{A, B, C\})=0.104
$$

Then, we can go through a similar process with comfort, fuel economy and style. We get the following information shown on Table 3.4.

Table 3.4: Priorities' values

| $C$ | $m_{C}$ | $S$ | $m_{S}$ | $F$ | $m_{F}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\{A\}$ | 0.526 | $\{A\}$ | 0.595 | $\{B\}$ | 0.833 |
| $\{B, C\}$ | 0.404 | $\{C\}$ | 0.277 | $\{A, B, C\}$ | 0.167 |
| $\{A, B, C\}$ | 0.07 | $\{A, B, C\}$ | 0.128 |  |  |

Then, these bba's must be combined with their criteria. Firstly, this step concerns the groups of criteria, that is the $\{C, S\}$ criteria. Our aim is to update the priority alternatives relative to the comfort and style criteria. Therefore, by using the Equation 3.8, we propose to combine the bba relative to the comfort and style criteria:

$$
m_{C, S}=m_{C} \odot m_{S}
$$

Then, these obtained bba's are discounted by the measure of reliability $\beta_{C, S}=0.17$ (see Table 3.2) and we use the Equation 3.6 and 3.7 to get the following Table 3.5.

Table 3.5: The bba $m_{C, S}$ after discounting

|  | $\emptyset$ | $\{A\}$ | $\{C\}$ | $\{B, C\}$ | $\Theta_{1}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $m_{C, S}^{\alpha_{C, S}}$ | 0.06564 | 0.07173 | 0.02232 | 0.00879 | 0.83152 |

After that, this step concerns the single criterion $\{P\}$ and $\{F\}$, the relative bba are directly discounting using the Equation 3.6 and 3.7, where the reliability measure $\beta_{P}=1$ and $\beta_{F}=0.55$. We get the following Table 3.6 and Table 3.7.

Table 3.6: The bba $m_{P}$ after discounting

|  | $\{A, B\}$ | $\Theta_{1}$ |
| :---: | :---: | :---: |
| $m_{P}^{\alpha_{P}}$ | 0.896 | 0.104 |

Table 3.7: The bba $m_{F}$ after discounting

|  | $\{B\}$ | $\Theta_{1}$ |
| :---: | :---: | :---: |
| $m_{F}^{\alpha_{F}}$ | 0.45815 | 0.54185 |

## Synthetic Utility

After updating the priorities of the alternatives sets with respect to their set of criteria, we must compute the overall bba. An intuitive definition of the strategy to calculate these bba's will be the conjunctive rule of combination generally used as an aggregate operator in the belief function framework combining between two or several bba's.

$$
\begin{equation*}
m_{\text {final }}=\odot m_{C_{k}}^{\alpha_{k}}, \quad k=\{1, \ldots, l\} \tag{3.9}
\end{equation*}
$$

where $l$ is the number of subsets of criteria.

## Example 3.5

Let us consider the same example. The conjunctive rule of combination is applied (Equation 3.9), this leads us to get a single bba denoted by $m_{\text {car }}$ (see Table 3.8).

Table 3.8: The bba $m_{\text {car }}$ after combining all evidences

|  | $\emptyset$ | $\{A\}$ | $\{B\}$ | $\{C\}$ | $\{A, B\}$ | $\{B, C\}$ | $\{A, C\}$ | $\Theta_{1}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $m_{\text {car }}$ | 0.1196 | 0.0389 | 0.3892 | 0.0013 | 0.4037 | 0.0005 | 0 | 0.0468 |

## Decision making

To this end, the final step is to choose the best alternative. In this context and through the use of the belief function theory which offers a process to help the expert to make a decision, we choose to use the pignistic transformation Equation 1.48 (Transformation from the power set to the frame of discernment). The decision maker will choose the alternative which has the highest value of pignistic probabilities.

## Example 3.6

The final step is then to choose the best alternatives; After computing the pignistic probabilities, we get:

$$
\operatorname{Bet} P_{c a r}(A)=0.2911, \operatorname{Bet} P_{c a r}(B)=0.6894 \text { and } \operatorname{Bet} P(C)_{c a r}=0.0195
$$

As a result, the car B will be preferred since it has the highest values.
Remark: By adopting the belief AHP method, our method provides some advantages to the decision maker: first, it is able to tolerate uncertainty and imprecision in our preferences. Second, the proposed approach has reduced the number of comparisons. In fact, if we have adopted the classical AHP, then there would be three comparisons per criterion between the decision alternatives level. That means, we will get 12 comparisons, and at the criterion level, we will have 6 comparisons. As a result, the number of pair-wise comparisons is then 18. However, by using the belief AHP, the number of comparisons decreases because instead of using single elements, we have used subsets.

### 3.6.3 The belief AHP procedure

The belief AHP procedure has the same skeleton as standard AHP method. So, to summarize the previous section, and to present the main steps of our
belief AHP method, we introduce its different construction steps, described as follows:

1. Model the problem as a hierarchy containing the decision goal, the sets of alternatives for reaching it, and the sets of criteria for evaluating the sets of alternatives.
2. Establish priorities among the elements of the hierarchy by making a series of judgments based on pair-wise comparisons of the elements.
3. For the criterion level: normalize the priorities' vectors and assume that each priority value is a measure of reliability for its corresponding subset of criteria.
4. For the alternative level, assume that each priority vector is a bba corresponding to a specific subsets of alternatives.
5. Synthesize the overall judgment, that is updating the sets of alternatives priorities with the importance of their corresponding criteria. Two cases will be presented: first, if we have a singleton criterion then Equation 3.6 and 3.7 are applied, else if we have a subset of criteria then Equation 3.8 is used.
6. Combine the overall bba's to get a single representation by using the conjunctive rule.
7. Come to a final decision based on the the pignistic transformation.

### 3.7 Belief pair-wise comparison matrix

Pair-wise comparisons aims at quantifying relative priorities for a given set of alternatives as well as the set of criteria, on a ratio scale, based on the judgment of the decision maker.

Using this approach, the decision maker has to express his opinion about the value of one single pair-wise comparison at a time. Usually, the decision maker has to choose his answer among discrete choices. Each choice is a linguistic phrase. Some examples of such linguistic phrases are: "A is more important than B ", or "A is of the same importance as B ", or "A is a little more important than $\mathrm{B} "$, and so on.

The decision maker uses the Saaty's scale (see Table 2.2) to map the labels which indicate the decision maker view to a numeric value.

However, as shown in (Joaquin, 1990) (Holder, 1995), this scale was criticized. With the Saaty's scale, the user cannot be consistent because it is not complete. Sometimes, the decision maker may well want to say that A is twice as important as $B$, and $A$ is 3 times as important as $C$, and $B$ is 1.5 times as important as C, yet he is constrained to make the last judgment 1 or 2 . In addition, the decision maker might find difficult to distinguish among them and tell for example whether one alternative is 6 or 7 times more important than another. Furthermore, the AHP method cannot cope with the fact that alternative A is 25 times more important than alternative C.

As a result, the scale is further incomplete and unnecessarily restricting because of the arbitrary cut-off at 9 for the maximum allowable ratio of weights.

Thus, our problem is as follows: how to quantify the linguistic choices selected by the decision maker during the evaluation of the pair-wise comparisons? Is it necessary to decompose even more the different levels of the Saaty's scale?

### 3.7.1 Acquisition and representation of uncertain decision knowledge

In this section, we propose to modify the structure of the Saaty's scale by adopting a new set of choices. Thus, to evaluate the responses of the pairwise comparison question, decision maker only selects the related linguistic variable. In fact, the expert only indicated whether a criterion was more or less important to its partner.

One aspect of the extended belief AHP method is the prevalence and allowance for uncertainty in the judgments made by decision maker. For instance, if a decision maker is unwilling or unable to specify a certain preference, then he is able to express his preferences with some degrees of uncertainty. Moreover, since the preferences in AHP are essentially human judgments based on knowledge and experience, the pair-wise comparison agrees well with the definition of these judgments with some levels of uncertainty.

In order to illustrate our proposed idea, we define for each alternative
(criteria) its own frame of discernment $\Theta$, which consists of the possible answers to the question: "Is alternative $A$ important?". To answer this question, the decision maker responds by "yes" or "no". In other terms:

$$
\begin{equation*}
\Theta_{A}=\{y e s, n o\} \tag{3.10}
\end{equation*}
$$

After identifying the frame of discernment $\Theta_{A}$ corresponding to the alternative $A$, our approach proposes to use a belief pair-wise comparison matrix. That is, each element of the matrix is described by a distributed assessment using a belief structure denoted by $m_{A}^{\Theta_{A}}($.$) . In other words, the decision$ maker can express his preferences with some degrees of uncertainty because in evaluation of qualitative criteria, for example, uncertain judgments could be used. For instance, in a problem of purchasing a car, the following type of uncertain subjective judgments was frequently used: "the comfort criterion is evaluated to be more important than style with a confidence degree of 0.6 " Thus, under each criterion or alternative, we will have a belief function.

To model the pair-wise matrix, some priorities must be respected. We consider $X$, the pair-wise comparison matrix, is an $n \times n$ matrix in which $n$ is the number of elements being compared. Entries of $X, d_{i j}$ 's are the judgments or the relative scale of alternative $i$ to alternative $j . d_{i j}$ is the entry from the $i^{\text {th }}$ column of $X$. It has the following characteristic:

1. $d_{i i}=m_{i}^{\Theta_{i}}\left(\Theta_{i}\right)=1$, where $\mathrm{i}=\mathrm{j}$.
2. If $d_{i j}=m_{j}^{\Theta_{i}}\left(\Theta_{i}\right)$, then $m_{i}^{\Theta_{j}}\left(\Theta_{j}\right)=d_{i j}$, where $i \neq j$
3. Otherwise, if $d_{i j}=m_{j}^{\Theta_{i}}$, then $d_{j i}=m_{i}^{\Theta_{j}}=\bar{m}_{j}^{\Theta_{i}}$, where $i \neq j$ and $\bar{m}$ is the negation of $m$.
The negation (or complement) $\bar{m}$ of a bba $m$ is defined as the bba verifying (Dubois \& Prade, 1986):

$$
\begin{equation*}
\bar{m}(A)=m(\bar{A}), \forall A \subseteq \Theta \tag{3.11}
\end{equation*}
$$

The same procedure is applied to compare criteria.

## Example 3.7

We assume there are three alternatives $A, B$ and $C$ that will be compared

Table 3.9: Belief pair-wise matrix

| $C_{1}$ | A | B | C |
| :---: | :---: | :---: | :---: |
| A | - $m_{A}^{\Theta_{A}}\left(\Theta_{A}\right)=1$ | $\begin{aligned} & -m_{B}^{\Theta_{A}}(\{y e s\})=0.6 \\ & -m_{B}^{\theta_{A}}(\{n o\})=0.3 \\ & -m_{B}^{\theta_{A}}\left(\Theta_{A}\right)=0.1 \\ & \hline \end{aligned}$ | $\begin{aligned} & -m_{C}^{\theta_{A}}(\{y e s\})=0.8 \\ & -m_{C}^{\theta_{A}}(\{n o\})=0.1 \\ & -m_{C}^{\theta_{A}}\left(\Theta_{A}\right)=0.1 \\ & \hline \end{aligned}$ |
| B | $\begin{aligned} & -m_{A}^{\theta_{B}}(\{y e s\})=0.3 \\ & -m_{A}^{\theta_{B}}(\{n o\})=0.6 \\ & -m_{A}^{\theta_{B}}\left(\Theta_{B}\right)=0.1 \\ & \hline \end{aligned}$ | - $m_{B}^{\Theta_{B}}\left(\Theta_{B}\right)=1$ | $\begin{aligned} & -m_{C}^{\theta_{B}}(\{\text { yes }\})=0.7 \\ & -m_{G}^{\theta_{B}}(\{n o\})=0.2 \\ & -m_{C}^{\theta_{B}}\left(\Theta_{B}\right)=0.1 \\ & \hline \end{aligned}$ |
| C | $\begin{aligned} & -m_{A}^{\theta_{C}}(\{\text { yes }\})=0.1 \\ & -m_{A}^{\theta_{C}}(\{n o\})=0.8 \\ & -m_{A}^{\theta_{C}}\left(\Theta_{C}\right)=0.1 \end{aligned}$ | $\begin{aligned} & -m_{B}^{\theta_{C}}(\{\text { yes }\})=0.2 \\ & -m_{B}^{\theta_{C}}(\{n o\})=0.7 \\ & -m_{B}^{\theta_{C}}\left(\Theta_{C}\right)=0.1 \end{aligned}$ | - $m_{C}^{\Theta_{C}}\left(\Theta_{C}\right)=1$ |

according to the criterion $c_{1}$ : we can get the following belief matrix (see Table 3.9).
where $m_{j}^{\Theta_{i}}$ represents the importance of the alternative $i$ relative to $j$ $(i, j=\{A, B, C\})$.

### 3.7.2 Partial combination

Once the pair-wise comparison matrix is complete, what is left is how to calculate the priority vector. In fact, within this structure of the comparison matrix, our belief AHP method cannot apply the eigenvector method to get the priority vector.

Thus, our problem is as follows: what is the appropriate function to use in order to obtain a single representation value of these different bba's to get the priority vector.

To perform this task, we have to define a rule for combining these bba's. In this context, we propose to regard each element belonging to the comparison matrix as a distinct source of information which provides distinct pieces of evidence. Then, based on the belief function framework, we can apply the conjunctive rule of combination to obtain a single representation value of these different bba's. Consequently, the obtained bba represents the importance of a specific alternative.

To better understand, we consider $X$, as defined above, the pair-wise comparison matrix. For each row of the matrix, we apply the conjunctive rule. That means, for each alternative $i(i=\{1 \ldots n\})$, we will get the following
bba:

$$
\begin{equation*}
m^{\Theta_{i}}=® m_{j}^{\Theta_{i}}, \text { where } j=\{1 \ldots n\} \tag{3.12}
\end{equation*}
$$

## Example 3.8

Let us continue with the previous example. For the alternative A, for example, we will get the following bba (see Table 3.10):

$$
m^{\Theta_{A}}=m_{A}^{\Theta_{A}} \bigcirc m_{B}^{\Theta_{A}} \bigcirc m_{C}^{\Theta_{A}}
$$

A similar procedure is repeated for the rest of alternatives, and we will get $m^{\Theta_{B}}$ and $m^{\Theta_{C}}$.

Table 3.10: Belief pair-wise matrix: Partial combination

| $C_{1}$ | Priority |
| :---: | :---: |
|  | $m^{\Theta_{A}}(\{$ yes $\})=0.62$ |
| A | $m^{\Theta_{A}}(\{n o\})=0.07$ |
|  | $m^{\Theta_{A}}(\emptyset)=0.3$ |
|  | $m^{\Theta_{A}}\left(\Theta_{A}\right)=0.01$ |
|  | $m^{\Theta_{B}}(\{y e s\})=0.31$ |
| B | $m^{\Theta_{B}}(\{n o\})=0.2$ |
|  | $m^{\Theta_{B}(\emptyset)=0.48}$ |
|  | $m^{\Theta_{B}}\left(\Theta_{B}\right)=0.01$ |
|  | $m^{\Theta_{C}}(\{y e s\})=0.05$ |
| $C$ | $m^{\Theta_{C}}(\{n o\})=0.71$ |
|  | $m^{\Theta_{C}(\emptyset)}=0.23$ |
|  | $m^{\Theta_{C}}\left(\Theta_{C}\right)=0.01$ |

### 3.7.3 Standardization of the frame of discernment

The main purpose of this stage is to standardize all the frames of discernment. In fact, when people use the same words, individual judgment of events is invariably subjective and may differ. In other terms, the problem here is that these introduced bba's are defined on different frames of discernment. Indeed, each alternative has its own frame of discernment. For example, if we say that alternative $A$ is more important than alternative $B$, and alternative $C$ is more important than alternative $B$, this does not mean that alternative $A$ and alternative $C$ have the same degree of importance.

In order to allow the combination of this information, we propose to use the concept of refinement operations (Shafer, 1976), which allows to establish relationships between different frames of discernment in order to express beliefs on anyone of them. The idea is then consists in obtaining one frame of discernment $\Theta$ from the set $\Theta_{k}$ by splitting some or all of its events.

So, each belief function $m^{\Theta_{k}}$ represents the belief over all possible answers (yes or no). However, at this stage, we want to know which alternative is the best $(\Theta=\{A, B, C\})$. As a result, $\Theta$ is considered as a a coarsening of $\Theta_{k}$, and we get the following relation:

$$
\begin{equation*}
m^{\Theta_{k} \uparrow \Theta}\left(\rho_{k}(A)\right)=m^{\Theta_{k}}(A), \quad \forall A \subseteq \Theta_{k} \tag{3.13}
\end{equation*}
$$

where the mapping $\rho_{k}$ from $\Theta_{k}$ to $\Theta$ is a refinement, and $\rho_{k}(\{y e s\})=\{k\}$ and $\rho_{k}(\{n o\})=\{\bar{k}\}$. After redefining the frame of discernment $\Theta$, what is left is updating the different bba's.

## Example 3.9

Let us continue with the previous example. For the alternative $A$ for example, if we apply the Equation 3.13 then we get the following bba:

$$
\begin{aligned}
& m^{\Theta_{A} \uparrow \Theta}(\{A\})=m^{\Theta_{A}}(\{y e s\}) \\
& m^{\Theta_{A} \uparrow \Theta}(\{\bar{A}\})=m^{\Theta_{A}}(\{n o\}) \\
& m^{\Theta_{A} \uparrow \Theta}(\{\Theta\})=m^{\Theta_{A}}\left(\Theta_{A}\right)
\end{aligned}
$$

To simplify, we can note by $m^{A, \Theta}$ the bba $m^{\Theta_{A} \uparrow \Theta}$, and the similar process is repeated for the rest of alternatives. We get the following matrix (see Table 3.11).

### 3.7.4 The overall combination

The aim of this step is to combine the obtained bba in order to compute the overall bba and to answer the question "What is the most important alternative?".

Table 3.11: Belief pair-wise matrix: Refinement

| $C_{1}$ | Priority |
| :---: | :---: |
|  | $m^{A, \Theta}(\{A\})=0.62$ |
| A | $m^{A, \Theta}(\{\bar{A}\})=0.07$ |
|  | $m^{A, \Theta}(\emptyset)=0.3$ |
|  | $m^{A, \Theta}(\Theta)=0.01$ |
|  | $m^{B, \Theta}(\{B\})=0.31$ |
| $B$ | $m^{B, \Theta}(\{\bar{B}\})=0.2$ |
|  | $m^{B, \Theta}(\emptyset)=0.48$ |
|  | $m^{B, \Theta}(\Theta)=0.01$ |
|  | $m^{C, \Theta}(\{C\})=0.05$ |
| C | $m^{C, \Theta}(\{\bar{C}\})=0.71$ |
|  | $m^{C, \Theta}(\emptyset)=0.23$ |
|  | $m^{C, \Theta}(\Theta)=0.01$ |

Once we have identified the same frame of discernment $\Theta$, and the obtained belief functions are expressed on the same frame of discernment, we can combine them using the conjunctive rule. So, we obtain a belief function reflecting the importance of alternatives to a given criterion. That is, we will apply the following rule:

$$
\begin{equation*}
m^{\Theta}=\oplus m^{i, \Theta} \text {, where } i=\{1 \ldots m\} \tag{3.14}
\end{equation*}
$$

where $m$ is the number of alternatives.

## Example 3.10

Let us continue with the previous example. The corresponding bba $m_{C 1}$ is defined by:

Table 3.12: The combined bba

|  | $\emptyset$ | $\{A\}$ | $\{B\}$ | $\{C\}$ | $\{A, B\}$ | $\{B, C\}$ | $\{A, C\}$ | $\Theta$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $m_{C 1}$ | 0.7472 | 0.096 | 0.00014 | 0.1565 | 0.00002 | 0.000007 | 0.000071 | 0.000001 |

## Remark

After obtaining the combined bba's, which reflect the importance of the alternatives with respect to a specific criterion, we notice that the same procedure used in the previous approach can be applied.

In fact, this proposed approach can be applied with the belief AHP method. So, instead of using the classical pair-wise comparison matrix, which is based on the Saaty's scale, the decision maker can use the belief pair-wise comparison and the main steps of the belief AHP approach can be then applied.

### 3.8 Conclusion

We have presented our method which consists in developing a new AHP method in an uncertain environment for MCDM methods. Our method is based on the belief function theory in order to represent the uncertainty relative to decision maker's preferences.

In this chapter, the belief AHP method was developed. We have exposed what kind of uncertainty is handled by this approach as well as its different objectives. Then, we have explained the main steps of our method.

Another part of the work provides some criticisms related to the comparison procedure, and we have proposed a more flexible method.

In the next chapter, we will present the implementation for checking the performance of our belief AHP method. Then, we will show the application of our approach on a real application problem.

## Chapter 4

## Implementation and application of the belief AHP method

### 4.1 Introduction

Implementing then applying our belief AHP approach are important since it allows us to have an idea concerning the flexibility of our proposed method.

In this chapter, we present the implementation of our new method, the belief Analytic Hierarchy Process (belief AHP) approach. To this end, we have developed all programs in MATLAB V7.4.

Once the different programs are implemented, for checking the flexibility and feasibility of our approach regarding belief AHP method and judging its performances, we have applied our proposed method on a real application problem. The problem considered here is the "PVC life cycle" especially its "end of life phase": the disposal problem.

Hence, this chapter is composed of two major parts. First, Section 4.2 deals with the implementation of belief AHP approach: we detail the major variables and programs used in our software. Then, we present the belief AHP algorithms. Second, Section 4.3 provides and analyzes the application of our proposed methods on a real world problem.

### 4.2 Implementation

### 4.2.1 Framework

In order to ensure the implementation of our approach, we have developed programs in Matlab V7.4.

As detailed in the previous chapter, these programs are developed to handle imprecision and uncertainty in two levels: the criteria and alternatives levels. In fact, our proposed method deals with groups of criteria and/or alternatives instead of single one.

These programs have as inputs:

1. The sets of different criteria.
2. The sets of different alternatives regarding each criterion.
3. The pair-wise comparison matrix which represents the preference of the decision maker by comparing all criteria, sub-criteria and alternatives with respect to upper level decision elements.

The outputs of our programs are mainly:

1. The different criteria with their corresponding weight.
2. The ranking of the alternatives according to the value of the pignistic probabilities, that is we will choose the alternative which has the highest pignistic probabilities.

### 4.2.2 Principal variables

The major variables that we have used in our programs to implement the belief AHP procedures are the following:

- criteria_matrix: a matrix whose first row contains the labels of the different sets of criteria, and in the remaining rows the corresponding values.
- data_criteria: a matrix that contains only the corresponding values for the sets of criteria.
- set_criteria: a vector that contains the different labels of criteria.
- vec_criteria: a vector that contains the priority vector.
- criteria: a record composed of:
- .label: the label of a given criterion or group of criteria.
- .value: the eigen value having that criterion label.
- alternative_ matrix: a matrix whose first row contains the labels of the different sets of alternatives, and in the remaining rows the corresponding values.
- data_alternative: a matrix whose contains only the corresponding values of each sets of alternatives.
- set_alternative: a vector that contains the different labels of alternatives.
- vec_alternative: a vector that contains the priority vector.
- alternative: a record composed of:
- .focal: the label of a given alternative (or a set of alternatives).
- .bba: the eigen value having that alternative label.
- .criterion: the label of the compared criterion.
- resultat: a record composed of:
- .focal: the label of a given alternative (or a set of alternatives).
- .bba: the resulting bba after discounting.
- inter: a record composed of:
- .focal: the label of a given alternative (or a set of alternatives).
- .bba: the combined bba's.
- nbre_alternative: the number of alternatives.
- number_criteria: the number of the sets of criteria.
- rule: a record composed of:
- .focal: the label of a given alternative (or a set of alternatives).
- .bba: the combined bba's.
- BetP: a record composed of:
- .focal: the label of a given alternative.
- .BetP: the pignistic probability for a given alternatives.


### 4.2.3 Belief AHP programs

In this subsection, we will present the major programs that we have developed to construct our software. We will regroup these programs according to the task they are used for.

## Pair-wise matrix

For each level of the hierarchy, we get sets of pair-wise comparisons matrices. In fact, to get the weight of criteria the relative pair-wise comparison is usually gained by expert via questionnaire. For the alternative level, the idea is to get for an incomplete decision matrix the pair-wise comparison of all the alternative regarding each criterion.

- Init_criteria: allows to load the information concerning sets of criteria.
- Init_alternative: allows to load the information concerning sets of alternatives by respecting each criterion.


## Belief AHP procedure

The implementation of "Belief_AHP" procedure relative to the belief AHP approach represents the important task. Many programs have been developed to ensure this purpose. Allowing the computation of different priorities values for each alternative and criterion to rank alternatives according to the value of the pignistic probabilities, and to find the best alternative. These main programs are defined as follows:

- Get_Preferences_Weights: is an iterative function allowing to compute the corresponding weight of each criterion and also groups of criteria relatively to the generated data from the loaded one. This procedure has as output "the info_criteria" record.
This Function uses the following main functions:
- Eig_Matrix: consists in computing the eigen vector of a given matrix.
- Normalize_Cr: normalizes the eigen vector of a given matrix to get the reliability degree.
- Get_Priority_Alternative: is an iterative function allowing to compute the priority vector for each pair-wise comparison matrix (alternative_ matrix).
- Eig_Matrix: it consists in computing the eigen vector of a given matrix.
- Normalize_Alt: normalizes the eigen vector of a given matrix.

Once the priorities' vector for each set of alternatives and the importance weight are generated, we have to develop programs that will ensure the updating of each sets of criteria with their respective importance of criterion. These programs are the following ones:

- Update_alternative: allows to discount the priority vector relative to each sets of alternatives according to the reliability degree of the followed criterion.
- Get_criteria: allows to get the label of the different criteria.
- Conjunctive_Rule: computes the conjunctive bba relative to a given matrix.
- Discount: allows to discount the relative bba's according to the reliability degree.
- Conjunctive_Rule: computes the conjunctive bba relative to a given matrix.
- Pignistic_function: computes the pignistic probability $B e t P$ relative to a given matrix.


## Belief pair-wise comparison

The implementation of "Belief_pair-wise comparison" procedure relative to the extended belief AHP approach represents the important task. Many programs have been developed to ensure this purpose. In fact, the main aim of this procedure is to compute the priorities values for each alternative and also criterion from a certain/uncertain preferences.

- Get_Belief_Priority: allows the computation of a certain/uncertain preferences from a belief decision matrix. This procedure uses the Conjunctive_Rule procedure to combine the conjunctive bba relative to a given matrix.


### 4.2.4 Belief AHP algorithms

In this section, we will present the major algorithms relative to the belief AHP procedure, namely, the belief AHP algorithm and the belief pair-wise comparison algorithm.

## Belief AHP

Input: criteria_matrix, alternative_matrix,criteria_tab, alternative_tab Output: BetP
. Begin
(* Load data set to criteria *)
[data_criteria] $\leftarrow$ Init_criteria(criteria_matrix)
4.
5. (* Getting the list of different sets of criteria and computing the relative weight *)
number_criteria $\leftarrow$ size(criteria, 2)
for $i=1$ to number_criteria do
set_criteria $\leftarrow[1:$ number_criteria $]$
vec_criteria $\leftarrow$ Get_Preferences_Weights(data_criteria)
criteria $(i)$.label $\leftarrow$ set_criteria
criteria $(i) . b b a \leftarrow$ vec_criteria
end for
13.
14. $\left(*\right.$ Load data set to alternatives (for each pair-wise matrix) ${ }^{*}$ )
15. [data_alternative] $\leftarrow$ Init_Alternative(alternative_matrix)
16.
17. (* For each comparison matrix: Getting the list of different sets of alternative and calculate the relative priority $*$ )
(*the number of criteria is the number of pair-wise comparison matrix*)
18. for $i=1$ to number_criteria do
set_alternative $\leftarrow[1$ : number_alternative $]$
vec_alternative $\leftarrow$ Get_Priority_Alternative(data_alternative)
criterion $\leftarrow$ Get_Criterion(Init_Alternative)
alternative $(i)$. focal $\leftarrow$ set_alternative
alternative $(i) . b b a \leftarrow v e c \_a l t e r n a t i v e$
alternative $(i)$.criterion $\leftarrow$ criterion
end for
26.
27. (* Initialization of useful variables *)
28. $k \leftarrow 1$
29. $l \leftarrow 1$
30. $j \leftarrow 0$
31. inter.focal $\leftarrow\}$
32. inter. $b b a \leftarrow[]$
33. resultat.focal $\leftarrow\}$
34. resultat.bba $\leftarrow[]$
35. nbre_alternative $\leftarrow$ size(alternative, 2 )
36.
37. (* Updating the alternatives priorities: two cases will be presentated: if the selected criterion is single or group of criteria *)
38. for $i=1$ to number_criteria do

```
    criterion \(\leftarrow\) criteria.label \((i)\)
    trouve \(\leftarrow\) false
    while (trouve \(\leftarrow\) false and \(\left.k \leq n b r e \_a l t e r n a t i v e\right) ~ d o\)
        \(j \leftarrow j+1\)
        if criterion \(==\) alternative \((j)\).criterion then
            var \(\leftarrow \operatorname{Discount(alternative~}(j)\), criteria.value \((i)\) )
            resultat \((k) \leftarrow\) var
            \(k \leftarrow k+1\)
            trouve \(\leftarrow\) true
        end if
        end while
        if trouve \(==\) false then
        for \(j=1\) to nbre_alternative do
            \(t m p=\) intersect(criterion, alternative \((j)\). criterion \()\)
            if \((t m p==\) alternative \((j)\).criterion \()\) then
```

54. $\quad$ inter $(l)$.focal $\leftarrow$ alternative $(j)$.focal
55. $\quad \operatorname{inter}(l) . b b a \leftarrow$ alternative $(j) . b b a$
56. $\quad l \leftarrow l+1$
57. end if
58. end for
59. $\quad$ comb $\leftarrow$ Conjunctive_Rule(inter)
60. $\quad$ temp $\leftarrow$ Discount (comb, criteria.value $(i))$
61. $\operatorname{resultat}(k) \leftarrow$ temp
62. $k \leftarrow k+1$
63. end if
64. end for
65. (* Combination of the groups of alternative *)
66. 
67. rule $\leftarrow$ Conjunctive_Rule(resultat)
68. (* Pignistic Probability *)
69. 
70. Bet $P \leftarrow$ pignistic (rule)
71. nbre_final $\leftarrow \operatorname{size}(\operatorname{Bet} P, 2)$
72. for $j=1$ to nbre_final do
73. print $\operatorname{BetP}(\mathrm{i})$.focal
74. print $\operatorname{BetP}(\mathrm{i})$. $\operatorname{BetP}$
75. end for
76. End

## Belief pair-wise matrix

Input: alternative_matrix, alternative_tab
Output: bba

1. Begin
2. (* Load data set to alternatives (for each pair-wise matrix) *)
3. [data_alternative] $\leftarrow$ Init_Alternative(alternative_matrix)
4. 
5. (* For each comparison matrix: Getting the list of different sets of alternative and calculate the overall bba *)
for $i=1$ to number_alternative do
6. for $j=1$ to number_alternative do
7. set_alternative $\leftarrow[1:$ number_alternative $]$
8. vec_alternative $\leftarrow$ Get_Preference(data_alternative)
9. alternative $(i, j)$.focal $\leftarrow$ set_alternative
10. alternative $(i, j) . b b a \leftarrow$ vec_alternative
```
        temp \(\leftarrow\) Conjunctive_Rule(alternative)
        end for
    \(b b a \leftarrow\) Conjunctive_Rule (temp)
end for
nbre_final \(\leftarrow \operatorname{size}(b b a, 2)\)
for \(i=1\) to \(n b r e \_\)final do
    print bba(i).focal
    print bba(i).bba
end for
```


### 4.3 Application

### 4.3.1 Introduction

In this section, the concepts presented in the previous chapter are applied on a real application problem.

The potential of this belief AHP method may be illustrated by considering a real application problem. Here, we consider "the PVC (Polyvinyl chloride) life cycle" especially "the end of life" phase of PVC. The problem considered here attempts to rank countries based on their environmental impact to the disposal of PVC product. Ideally, the rank of a country for attention must be judged over a number of different environmental criteria.

The main aim is then to demonstrate an applicable way of improving evaluation tactics in complex decision problems, where there is a large number of criteria and there is a need to follow human behavior. So, uncertainty comes closer to reality compared to classical evaluation processes using crisp data.

The PVC life cycle is not the main focus of this study. The focus is to demonstrate the principal feasibility of the belief AHP approach in combination with a real application problem.

The application of the belief AHP method to the data from the PVC disposal case study is described in the following section.

### 4.3.2 PVC disposal

## Introduction

Polyvinyl chloride (PVC) is one of the three most important polymers currently used worldwide. This is because PVC is one of the cheapest polymers to make and it has a large range of properties. So, it can be used to make hundreds of products.

PVC has become widespread among our daily lives and industrial activities in the form of various products. For such reasons, its safety and impact on human health has become the center of concern by the general public.

The aim of this part of the report is to discuss the life cycle impacts of PVC polymer. So, the PVC life cycle can be structured in different phases and it is divided into three phases: the production phase (of the main components of PVC products, including raw materials), the use phase and the end-of-life phase. This application covers only "the end of life" phase of PVC. This final stage of PVC's life cycle creates the most severe environmental hazards.

From the processing of its raw materials through to its disposal, PVC creates environmental and human health problems. The origins of these problems lie in the properties of the toxic chemicals that are used to make PVC product. So, many PVC products contain additional chemicals to change the chemical consistency of the product.

In fact, after its useful life, the vinyl product is disposed of, typically in incinerators or landfills. Environmental impacts at this stage include the long-term persistence of vinyl products in land disposal facilities, the product being leached of hazardous substances. As a result, many negative effects were appeared from burning or landfilling products containing PVC plastics. Dioxins and furans are released when they are burnt. When landfilled, they contaminate groundwater because of the leaching of toxic additives in the PVCs and contaminate the air because of toxic emissions in landfill gases.

## Problematic

There is a growing concern regarding the potential human health and environmental impacts of producing, using and disposing of PVC-containing
materials. That is why, the problem here is to use or not the PVC in general, but to know in which country the environmental impact is less important for the destruction of a kilogram of PVC?

Especifically and in this context, the challenge facing an expert here is the choice of the country where the environmental impact is the least important for the destruction of a kilogram of PVC.

As a result, this problem can be considered as a multi-criteria decision making problem. In this decision making problem, environmental criteria are playing the role of multiple criteria, which are the crucial keys to solving this problem, and "Switzerland, France, USA or England" are the set of alternatives.

So, classical AHP is thought to be a robust way to solve determined decision making problem. However, it neglects the uncertainty and imprecision caused by subjective preference of decision maker in criteria and alternative scoring. Accordingly, belief AHP was used to improve this situation.

Problem comes out naturally that whether or not belief AHP is appropriate to solve this problem. The applicability of our proposed method to this specified problem is justified in the next section.

### 4.3.3 Identification of the PVC production problem

This section presents the details of the PVC disposal problem investigated throughout this study. Firstly, we will apply the belief AHP approach introduced in the previous chapter (Section 3.6). Then, we will use the belief AHP method with the belief pair-wise comparison (Section 3.7).

## Belief AHP approach

In this application problem, an expert in the area of PVC life cycle assessment was asked to take part in the study. The decision maker was familiar with the PVC disposal environment.

Within belief AHP approach, there are two levels of judgment making to be undertaken by each of the decision maker. Firstly, a weighting on the importance of each criterion needs to be expressed. Secondly, the preference judgments on the countries over each of the criteria were needed to be made.

## - The candiadate criteria:

The first stage was the identification of the necessary criteria to be considered, which here was a consequence of a semi-structured interview with the expert. So, the selection of these appropriate criteria is an important step.
Following discussion with the expert concerning the nature of the application, it was decided to restrict the number of criteria to ten areas:

1. Abiotic depletion (C1): reflects the consumption of minerals and fossil resources, calculated in kg antimony equivalent, based on available reserves.
2. Acidification (C2): the gradual decrease in the pH of the oceans.
3. Eutrophication (C3): refers to natural or artificial addition of nutrients to bodies of water and to the effects of the added nutrients. When the effects are undesirable, eutrophication may be considered a form of pollution.
4. Ozone layer depletion steady state (C4): Depletion of the ozone layer method developed by WMO (World Meteorological Organization).
5. Human toxicity infinite (C5): this category concerns effects of toxic substances on the human environment.
6. Fresh water aquatic ecotoxicity infinite (C6): this category indicator refers to the impact on fresh water ecosystems, as a result of emissions of toxic substances to air, water and soil.
7. Terrestrial ecotoxicity infinite (C7): this category refers to impacts of toxic substances on terrestrial ecosystems.
8. Photochemical oxidation (C8): is the formation of reactive substances (mainly ozone) which are injurious to human health and ecosystems and which also may damage crops.
9. IPCC (C9): Explains the Greenhouse Effect. The result of water vapor, carbon dioxide, and other atmospheric gases trapping radiant (infrared) energy, thereby keeping the earth's surface warmer than it would otherwise be.
10. Cumulative energy demand (C10): is a method for calculating the environmental impacts of products and services throughout their entire life cycle.

Given the necessary details of the criteria, the expert was asked to indicate an order of preferences between criteria through the structured interview. One characteristic of this questionnaire was that it stated that the expert could give his preferences on groups of criteria. The questionnaire was designed to avoid pressuring the expert into an inappropriate decision by allowing comparisons between subsets of criteria and not compulsory on singleton criterion. This advantage is one facet of the utilization of belief AHP method.

Then, for each criterion the decision maker indicates his level of preference. Importantly, the decision maker was made aware that when assigning the same scale values to criteria, he was grouping them together in subsequent analysis. The judgments made by the decision maker are reported in Table 4.1.

Table 4.1: The weights assigned to the criteria according to the expert's opinion

| Criteria | $\{C 5\}$ | $\{C 6\}$ | $\{C 1, C 3, C 7\}$ | $\{C 2, C 4, C 8, C 9, C 10\}$ |
| :---: | :---: | :---: | :---: | :---: |
| $\{C 5\}$ | 1 | 4 | 6 | 9 |
| $\{C 6\}$ | $\frac{1}{4}$ | 1 | $\frac{1}{4}$ | 6 |
| $\{C 1, C 3, C 7\}$ | $\frac{1}{6}$ | 4 | 1 | $\frac{1}{3}$ |
| $\{C 2, C 4, C 8, C 9, C 10\}$ | $\frac{1}{9}$ | $\frac{1}{6}$ | 3 | 1 |

From Table 4.1, we conclude that the decision maker has identified four subsets of criteria $\{C 5\},\{C 6\},\{C 1, C 3, C 7\}$, and $\{C 2, C 4, C 8, C 9, C 10\}$. In fact, criteria which belong to the same subsets have the same degree of preferences, that means criteria which have the same value scale are grouped together. Thus, we could suppose that these criteria have the same importance and consequently have the same distributed weights.
By grouping the criteria in subsets of criteria, the decision maker has reduced the number of pair-wise comparison. Indeed, by adopting standard AHP, the decision maker has to make $\frac{m(m-1)}{2}$ pair-wise comparisons with $m$ is the number of criteria. However, by using the belief AHP approach, in the worst case, the decision maker makes $\frac{m(m-1)}{2}$ pair-wise comparisons (when he compares singleton criterion between each other). Otherwise, if we consider $p$ is the number of subsets of criteria, the decision maker needs only $\frac{p(p-1)}{2}$ pair-wise comparisons, where $p<m$. The number of subsets $p$ depends on the judgments made by the expert whether to include a criterion in a particular group of criteria or not.

In our case for example, if we use the standard AHP, we need to make 45 pair-wise comparisons. With belief AHP, we only make 6 pair-wise comparisons.

## - The candiadate alternatives:

Apart from the ten criteria, the initial interview also identified four selected countries: "Switzerland (SW), France (FR), USA (US) and England (ENG)" which are the set of alternatives. We would generate a decision hierarchy by which it is possible to evaluate different alternatives.

The next stage within belief AHP is then to make judgments on the alternatives over the different criteria. Judgments on the identified countries are with respect to all considered alternatives (the frame of discernment). To differentiate between the preferences on the groups of alternatives identified by a decision maker, a set of verbal preference scales are utilized with an associated set of numerical scale values. That means, we followed the suggestion of Saaty method (Saaty's scale).
For this purpose, we take the quantitative data from SIMAPRO software (Ecoinvent database), on the basis of which we evaluate the four identified alternatives. In this respect, it has to be mentioned, that data collection was done by a simple, spreadsheet-based computer program. Importantly, the decision maker was made aware that when assigning the same scale values to decision alternatives, he was grouping them together in subsequent analysis.
To illustrate this, for the criteria C1 (Abiotic depletion), we get two sets of alternatives $\{E N G, U S\}$ and $\{F R\}$ (see Table 4.2). All the alternatives in the same group are considered to be of equal preference. Hence, no two groups identified can have the same scale values assigned to them, since if they had they should be combined into a single group.

The inability of the decision maker to make a preference judgment on the alternative $\{S W\}$ (Switzerland) may be due to: that no information/knowledge exists for the decision maker to make a judgment with respect to the "Abiotic depletion" criterion ( $C 1$ ). This is what makes up the uncertainty inherent in the decision making process, underlying the advantage of utilizing belief AHP. So, each group of alternatives identified is compared to the frame of discernment $\Theta$ to allow for the opportunity of the allocation of ignorance on the knowledge of evidence. This group of alternatives $\Theta$ is then used to express the uncertainty/imprecision when, for example, the decision maker is not able to express his preferences on a particular alternative.

Similarly to the criterion level, the judgments between decision alternatives over different criteria are dealt within an identical manner. In our case, we get the following pair-wise comparisons matrices:

Table 4.2: Comparison matrix regarding C 1 criterion

| $C 1$ | $\{E N G, U S\}$ | $\{F R\}$ | $\Theta$ |
| :---: | :---: | :---: | :---: |
| $\{E N G, U S\}$ | 1 | $\frac{1}{2}$ | 4 |
| $\{F R\}$ | 2 | 1 | 2 |
| $\Theta$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 |

Table 4.3: Comparison matrix regarding C 2 criterion

| $C 2$ | $\{E N G, U S\}$ | $\{F R\}$ | $\Theta$ |
| :---: | :---: | :---: | :---: |
| $\{E N G, U S\}$ | 1 | $\frac{1}{2}$ | 4 |
| $\{F R\}$ | 2 | 1 | 2 |
| $\Theta$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 |

Table 4.4: Comparison matrix regarding C3 criterion

| $C 3$ | $\{F R, S\}$ | $\Theta$ |
| :---: | :---: | :---: |
| $\{F R, S\}$ | 1 | 2 |
| $\Theta$ | $\frac{1}{2}$ | 1 |

Table 4.5: Comparison matrix regarding C 4 criterion

| $C 4$ | $\{E N G, U S\}$ | $\{F R\}$ | $\Theta$ |
| :---: | :---: | :---: | :---: |
| $\{E N G, U S\}$ | 1 | $\frac{1}{2}$ | 4 |
| $\{F R\}$ | 2 | 1 | 2 |
| $\Theta$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 |

Table 4.6: Comparison matrix regarding C5 criterion

| $C 5$ | $\{F R, U S, E N G\}$ | $\Theta$ |
| :---: | :---: | :---: |
| $\{F R, U S, E N G\}$ | 1 | 2 |
| $\Theta$ | $\frac{1}{2}$ | 1 |

Table 4.7: Comparison matrix regarding C6 criterion

| $C 6$ | $\{F R, S W\}$ | $\Theta$ |
| :---: | :---: | :---: |
| $\{F R, S W\}$ | 1 | 2 |
| $\Theta$ | $\frac{1}{2}$ | 1 |

Table 4.8: Comparison matrix regarding C 7 criterion

| $C 7$ | $\{E N G, U S\}$ | $\{F R\}$ | $\Theta$ |
| :---: | :---: | :---: | :---: |
| $\{E N G, U S\}$ | 1 | $\frac{1}{2}$ | 4 |
| $\{F R\}$ | 2 | 1 | 2 |
| $\Theta$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 |

Table 4.9: Comparison matrix regarding C 8 criterion

| $C 8$ | $\{E N G, U S\}$ | $\{F R\}$ | $\Theta$ |
| :---: | :---: | :---: | :---: |
| $\{E N G, U S\}$ | 1 | $\frac{1}{2}$ | 4 |
| $\{F R\}$ | 2 | 1 | 2 |
| $\Theta$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 |

Table 4.10: Comparison matrix regarding C9 criterion

| $C 9$ | $\{E N G, U S\}$ | $\{F R\}$ | $\Theta$ |
| :---: | :---: | :---: | :---: |
| $\{E N G, U S\}$ | 1 | $\frac{1}{2}$ | 4 |
| $\{F R\}$ | 2 | 1 | 2 |
| $\Theta$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 |

Table 4.11: Comparison matrix regarding C10 criterion

| $C 10$ | $\{E N G, U S\}$ | $\{F R\}$ | $\Theta$ |
| :---: | :---: | :---: | :---: |
| $\{E N G, U S\}$ | 1 | $\frac{1}{2}$ | 4 |
| $\{F R\}$ | 2 | 1 | 2 |
| $\Theta$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 |

## - Results:

In fact, after identifying the sets of criteria and also the sets of alternatives and their corresponding preferences, we can now use our implemented software to get the final rank of the identified alternatives. So, the main steps of the belief AHP method are already defined in the previous chapter. Out of the pair-wise comparison matrix, we
can follow these main stages:

1. Computing the weight of considered criteria: $\{C 5\},\{C 6\},\{C 1, C 3, C 7\}$, and $\{C 2, C 4, C 8, C 9, C 10\}$ using the eigen vector method. Then, we normalize this eigen vector, and we assume that each eigen value corresponding to a particular subsets of criteria is considered as a measure of reliability.
2. Computing the alternatives' priorities using the eigen vector method, and transforming this eigen vector into a bba. As a result, we get ten bba's relative to each criterion.
3. Updating the alternatives priorities. First, the reliability measure of the criterion $\{C 5\}$ and $\{C 6\}$ is directly discounted with their corresponding bba's. Then, for the subsets of criteria $\{C 1, C 3, C 7\}$ and $\{C 2, C 4, C 8, C 9, C 10\}$, we have combined their corresponding bba's using the conjunctive rule. After that, the obtained bba is discounted by the measure of reliability relative to the specific group of criteria.
4. Combine the overall bba's to get a single representation by using the conjunctive rule.
5. Ranking of the alternatives according to the pignistic transformation.

Furthermore, in this Section, we have identified the structure of the proposed problem in order to show the feasibility of our proposed approach on a real application problem.

By using our implemented software, we get this final result (see Table 4.12):

Table 4.12: The final result using the belief AHP approach

|  | Belief AHP |
| :---: | :---: |
| ENG | 0.088249 |
| FR | 0.53811 |
| USA | 0.087038 |
| SW | 0.2866 |

The decision maker wants to know in which country the environmental impact is less important for the destruction of a kilogram of PVC.

The end results using belief AHP process are shown in Table 4.12. The table shows that alternative $\{F R\}$ has the highest pignistic probabilities 0.53811 . Consequently, " France "is the recommended country since it has the highest values.
In order to compare our belief AHP approach with the standard version of AHP method, we propose to convert all the pair-wise matrix in the standard form. That is for criterion level, we have converted the pairwise comparison matrix from $4 \times 4$ matrix into $10 \times 10$, where 4 is the number of subsets of criteria and 10 is the number of criteria. A similar process is also repeated for all the pair-wise matrix which compare the alternatives according to each criterion. Then, for each criterion, we get $4 \times 4$ matrix. After that, we have applied the Saaty approach and we have got the following result (see Table 4.13):

Table 4.13: The final ranking using standard AHP method

|  | Standard AHP |
| :---: | :---: |
| ENG | 0.1654 |
| FR | 0.502 |
| USA | 0.0906 |
| SW | 0.242 |

By the use of belief AHP method, there is a slight shift from less important to more important alternatives. Thus, alternatives have kept the same ranking.
This research proposes that the differences between classic AHP method and belief AHP approach are raised by two potential factors. One, belief AHP method can provide the decision maker to express his preferences with some uncertainty and imprecision rather than deterministic value options. This gives an advantage over classic AHP method in solving complex problems. The use of subsets of criteria and/or alternatives for pair-wise comparison is hence the first factor. Two, by using belief AHP method the number of comparison is reduced (see Table 4.14).

Table 4.14: Comparison between standard AHP and belief AHP

|  | Standard AHP | Belief AHP |
| :---: | :---: | :---: |
| Criteria | $\frac{m(m-1)}{2}$ | $\frac{p(p-1)}{2}$ |
| Alternatives | $\frac{m \cdot n(n-1)}{2}$ | $\frac{m \cdot k(k-1)}{2}$ |
| Total | $\frac{m}{2}\left(n^{2}-n+m-1\right)$ | $\frac{1}{2}\left(m \cdot k^{2}-m \cdot k+p^{2}-p\right)$ |

Where:

- $m$ and $n$ are respectively the number of criteria and alternatives.
- $p$ and $k$ are respectively the number of subsets of criteria and alternatives.
- $m \succeq p, p$ is equal to $m$ when the decision maker compares singleton criterion between each other.
$-n \succeq k, k$ is equal to $n$ when the decision maker compares singleton alternative between each other.

In our case, if we have adopted the classical AHP, then there would be 6 comparisons per criterion between the decision alternatives level. That means, we will get 60 comparisons, and at the criterion level, we will have 45 comparisons. As a result, the number of pair-wise comparisons is then 105 . However, by using the belief AHP, the number of comparisons decreases because instead of using single elements, we have used subsets. In fact, in our case the decision maker had made 6 pair-wise comparisons at the criterion level, and at the alternative level, he had made 24 pair-wise comparisons.
Furthermore, regarding to the computational complexity, our proposed approach reduces the complexity in term of pair-wise comparison. In fact, the number of levels in the hierarchy depends on the complexity of the decision problem. However, looking into the AHP methodology, the number of pair-wise comparisons increases exponentially with each additional criterion. So, the theoretical complexity of this method is then $O\left(m^{2}\right)$, here $m$ is the number of criteria. In the worst case, our belief AHP method has the same computational complexity as standard AHP approach.

## Belief AHP approach with belief pair-wise comparison

Though the popularity and efficiency of the AHP approach, this method is often criticized, because it cannot faithfully represent decision maker's
preferences given by a numerical representation of his judgments.
The only difference between this proposed approach and the belief AHP method is the use of the pair-wise comparisons procedure. In fact, with belief pair-wise comparison, the decision maker will be able to introduce uncertainty in the alternative or criterion judgments. Instead of using crisp number to express the preferences, the decision maker will be able to make uncertain judgment.

Let us consider our application problem with the same sets of criteria and alternatives. We will consider the same weight of criteria obtained with the belief AHP method, and we will apply the belief pair-wise comparaison on the alternative level.

Using the belief pair-wise comparison, the decision maker does not need to introduce the set $\Theta$ to express his uncertainty. Thus, to evaluate the responses of the pair-wise comparison question, he will be able to express his judgment with some degrees of uncertainty. That's, each element of the matrix will be labeled by a bba or mass function expressing a belief on judgment value. So, the question is how construct these bba's to obtain uncertain preferences.

For this purpose, several interviews with the expert were realized in order to model these bba's. Consequently and after these discussions, the expert has validated the following procedure in order to create these bba's.

Indeed, the resulting bba's has 2 focal elements:

- The first is the actual preference's value regarding the alternative $A$ with $\operatorname{bbm}, m(A)=p * 0.1$ (In our case $p$ is obtained from the classical Saaty's scale comparaison).
- The second is $\Theta$ such as $m(\Theta)=1-m(A)$

As a result, the expert has validated the following belief pair-wise matrices (see Table 4.15 to Table 4.24). For example, to evaluate the alternatives according to the criterion $C 1$ (Abiotic depletion), the decision maker is asked to evaluate the following subsets of alternatives: $\{E N G, U S\},\{F R\}$ and $\{S W\}$ according to the criterion $C 1$. For instance, he may say that $\{F R\}$ is evaluated to be more important than $\{E N G, U S\}$ with a confidence degree of 0.2 . That means, 0.2 of beliefs are exactly committed to the alternative $\{F R\}$ is more important than $\{E N G, U S\}$, whereas 0.8 is assigned to the
whole frame of discernment (ignorance). We get the following matrix (see Table 4.15)

Table 4.15: Belief pair-wise matrix regarding C 1 criterion

| C1 | \{ENG,US\} | $\{F R\}$ | \{SW\} |
| :---: | :---: | :---: | :---: |
| \{ENG, US \} | $m^{\theta_{\{E N G, U S\}}( }\left(\Theta_{\{E N G, U S\}}\right)=1$ | $\begin{aligned} & m^{\left.\Theta_{\{E N G, U S\}}\right\}}(\{n o\})=0.05 \\ & m^{\Theta_{\{E N G, U S\}}}\left(\Theta_{\{E N G, U S\}}\right)= \\ & 0.95 \end{aligned}$ | $\begin{aligned} & m^{\Theta_{\{E N G, U S\}}(\{y e s\})=0.4} \\ & m^{\Theta_{\{E N G, U S\}}}\left(\Theta_{\{E N G, U S\}}\right)= \\ & 0.6 \end{aligned}$ |
| \{FR\} | $\begin{aligned} & m^{\Theta_{\{F R\}}}(\{y e s\})=0.05 \\ & -m^{\theta_{\{F R\}}}\left(\Theta_{\{F R\}}\right)=0.95 \end{aligned}$ | $m^{\Theta_{\{F R\}}( }\left(\Theta_{\{F R\}}\right)=1$ | $\begin{gathered} \left.m^{\theta_{\{F R\}}(\{y e s\}}\right)=0.2 \\ -m^{\Theta_{\{F R\}}}\left(\Theta_{\{F R\}}\right)=0.8 \\ \hline \end{gathered}$ |
| \{SW\} | $\begin{aligned} & m^{\Theta_{\{S W\}}(\{n o\})}=0.4 \\ & -m^{\Theta_{\{S W\}}}\left(\Theta_{\{S W\}}\right)=0.6 \end{aligned}$ | $\begin{aligned} & m^{\Theta_{\{S W\}}(\{n o\})=0.2} \\ & -m^{\Theta_{\{S W\}}}\left(\Theta_{\{S W\}}\right)=0.8 \end{aligned}$ | $m^{\left.\theta_{\{S W}\right\}}\left(\Theta_{\{S W\}}\right)=1$ |

Similarly, the belief pair-wise matrixes among the four alternatives towards the ten criteria are computed below in the following tables.

Table 4.16: Belief pair-wise matrix regarding C2 criterion

| C2 | $\{E N G, U S\}$ | \{FR\} | $\{S W\}$ |
| :---: | :---: | :---: | :---: |
| $\{E N G, U S\}$ | $m^{\Theta\{E N G, U S\}}$ ( $\left.\Theta_{\{E N G, U S\}}\right)=1$ | $\begin{aligned} & m^{\Theta_{\{E N G, U S\}}(\{n o\})=0.05} \\ & m^{\Theta_{\{E N G, U S\}}}\left(\Theta_{\{E N G, U S\}}\right)= \\ & 0.95 \end{aligned}$ |  |
| \{FR\} | $\begin{aligned} & m^{\Theta_{\{F R\}}(\{y e s\})}=0.05 \\ & -m^{\Theta_{\{F R\}}}\left(\Theta_{\{F R\}}\right)=0.95 \end{aligned}$ | $\left.m^{\Theta_{\{F R\}}( } \Theta_{\{F R\}}\right)=1$ | $\begin{aligned} & m^{\Theta_{\{F R\}}(\{y e s\})}=0.2 \\ & -m^{\Theta_{\{F R\}}}\left(\Theta_{\{F R\}}\right)=0.8 \end{aligned}$ |
| \{SW\} | $\begin{aligned} & m^{\Theta_{\{S W\}}(\{n o\})=0.4} \\ & -m^{\Theta_{\{S W\}}}\left(\Theta_{\{S W\}}\right)=0.6 \end{aligned}$ | $\begin{aligned} & m^{\Theta_{\{S W\}}(\{n o\})=0.2} \\ & -m^{\Theta_{\{S W\}}}\left(\Theta_{\{S W\}}\right)=0.8 \end{aligned}$ | $m^{\Theta_{\{S W}( }\left(\Theta_{\{S W\}}\right)=1$ |

Table 4.17: Belief pair-wise matrix regarding C3 criterion

| C3 | $\{F R, S W\}$ | $\{E N G\}$ | $\{U S\}$ |
| :---: | :---: | :---: | :---: |
| $\{F R, S W\}$ | $\left.m^{\Theta_{\{F R, S W\}}( } \Theta_{\{F R, S W\}}\right)=1$ | $\begin{aligned} & m^{\Theta_{\{F R, S W\}}}(\{n o\})=0.2 \\ & m^{\Theta_{\{F R, S W\}}}\left(\Theta_{\{F R, S W\}}\right)=0.8 \end{aligned}$ | $\begin{aligned} & m^{\Theta_{\{F R, S W\}}(\{y e s\})=0.2} \\ & m^{\Theta_{\{F R, S W\}}}\left(\Theta_{\{E N G, U S\}}\right)= \\ & 0.8 \end{aligned}$ |
| \{ENG\} | $\begin{aligned} & m^{\Theta_{\{E N G\}}(\{y e s\})=0.2} \\ & -m^{\Theta_{\{E N G\}}}\left(\Theta_{\{E N G\}}\right)=0.8 \end{aligned}$ | $m^{\Theta}{ }_{\{E N G\}}\left(\Theta_{\{E N G\}}\right)=1$ | $\begin{aligned} & m^{\Theta_{\{E N G\}}}(\{\text { yes }\})=0.05 \\ & -m^{\Theta_{\{E N G\}}}\left(\Theta_{\{E N G\}}\right)=0.95 \end{aligned}$ |
| \{US |  | $\begin{aligned} & m^{\Theta_{\{U S\}}}(\{n o\})=0.05 \\ & -m^{\Theta_{\{U S\}}}\left(\Theta_{\{U S\}}\right)=0.95 \\ & \hline \end{aligned}$ | $\left.m^{\Theta_{\{U S\}}( } \Theta_{\{U S\}}\right)=1$ |

Table 4.18: Belief pair-wise matrix regarding C 4 criterion

| C4 | $\{E N G, U S\}$ | \{FR\} | $\{S W\}$ |
| :---: | :---: | :---: | :---: |
| $\{E N G, U S\}$ | $\left.m^{\Theta_{\{E N G, U S\}}( } \Theta_{\{E N G, U S\}}\right)=1$ | $\begin{aligned} & m^{\Theta_{\{E N G, U S\}}(\{n o\})=0.05} \\ & m^{\Theta_{\{E N G, U S\}}}\left(\Theta_{\{E N G, U S\}}\right)= \\ & 0.95 \end{aligned}$ | $\left.\begin{array}{l} m^{\Theta} \Theta_{\{E N G, U S\}}(\{\text { yes }\})=0.4 \\ m^{\Theta \in E N G, U S\}}( \end{array} \Theta_{\{E N G, U S\}}\right)=$ |
| \{FR\} | $\begin{aligned} & m^{\Theta_{\{F R\}}(\{y e s\})}=0.05 \\ & m^{\Theta_{\{F R\}}}\left(\Theta_{\{F R\}}\right)=0.95 \end{aligned}$ | $\left.m^{\Theta_{\{F R\}}( } \Theta_{\{F R\}}\right)=1$ | $\begin{aligned} & m^{\Theta_{\{F R\}}(\{\text { yes }\})}=0.8 \\ & -m^{\Theta_{\{F R\}}}\left(\Theta_{\{F R\}}\right)=0.2 \end{aligned}$ |
| \{SW\} | $\begin{aligned} & m^{\Theta_{\{S W\}}(\{n o\})=0.4} \\ & -m^{\Theta_{\{S W\}}}\left(\Theta_{\{S W\}}\right)=0.6 \end{aligned}$ | $\begin{aligned} & m^{\Theta_{\{S W\}}(\{n o\})=0.8} \\ & -m^{\Theta_{\{S W\}}}\left(\Theta_{\{S W\}}\right)=0.2 \end{aligned}$ | $m^{\left.\Theta_{\{S W}\right\}}\left(\Theta_{\{S W\}}\right)=1$ |

Table 4.19: Belief pair-wise matrix regarding C5 criterion

| $C 5$ | $\{F R, E N G, U S\}$ | $\{S W\}$ |
| :---: | :---: | :---: |
| $\{F R, E N G, U S\}$ | $m^{\Theta} \Theta_{\{F R, E N G, U S\}}\left(\Theta_{\{F R, E N G, U S\}}\right)=1$ | $\begin{aligned} & m^{\Theta_{\{F R, E N G, U S\}}(\{y e s\})=0.2} \\ & m_{\{F R, E N G, U S\}}\left(\Theta_{\{F R, E N G, U S\}}\right)= \\ & 0.8 \end{aligned}$ |
| \{SW\} | $\begin{aligned} & m^{\Theta_{\{S W\}}(\{n o\})=0.2} \\ & m^{\Theta_{\{S W\}}}\left(\Theta_{\{F R\}}\right)=0.8 \end{aligned}$ | $m^{\Theta_{\{S W\}}}\left(\Theta_{\{S W\}}\right)=1$ |

Table 4.20: Belief pair-wise matrix regarding C6 criterion

| C6 | \{FR, SW \} | \{ENG\} | \{US\} |
| :---: | :---: | :---: | :---: |
| \{FR, SW \} | $m^{\theta_{\{F R, S W\}}\left(\Theta_{\{F R, S W\}}\right)=1}$ | $\begin{aligned} & m^{\Theta_{\{F R, S W\}}(\{n o\})=0.2} \\ & m^{\Theta_{\{F R, S W\}}}\left(\Theta_{\{F R, S W\}}\right)=0.8 \end{aligned}$ | $\begin{aligned} & m^{\theta_{\{F R, S W\}}}(\{y e s\})=0.2 \\ & m^{\Theta_{\{F R, S W\}}\left(\Theta_{\{E N G, U S\}}\right)}= \\ & 0.8 \end{aligned}$ |
| \{ENG\} | $\begin{aligned} & m^{\theta_{\{E N G\}}}(\{y e s\})=0.2 \\ & -m^{\theta_{\{E N G\}}}\left(\Theta_{\{E N G\}}\right)=0.8 \end{aligned}$ | $m^{\Theta_{\{E N G\}}\left(\Theta_{\{E N G\}}\right)=1}$ | $\begin{aligned} & m^{\Theta_{\{E N G\}}(\{y e s\})=0.05} \\ & -m^{\Theta_{\{E N G\}}}\left(\Theta_{\{E N G\}}\right)=0.95 \end{aligned}$ |
| \{US\} | $\begin{aligned} & m^{\Theta_{\{U S\}}}(\{n o\})=0.2 \\ & -m^{\Theta_{\{U S\}}}\left(\Theta_{\{U S\}}\right)=0.8 \end{aligned}$ | $\begin{aligned} & m^{\Theta_{\{U S\}}}(\{n o\})=0.05 \\ & -m^{\Theta_{\{U S\}}}\left(\Theta_{\{U S\}}\right)=0.95 \end{aligned}$ | $m^{\Theta_{\{U S\}}}\left(\Theta_{\{U S\}}\right)=1$ |

Table 4.21: Belief pair-wise matrix regarding C7 criterion

| $C 7$ | $\{E N G, U S\}$ | $\{F R\}$ | $\{S W\}$ |
| :---: | :---: | :---: | :---: |
| $\{E N G, U S\}$ | $\left.m^{\Theta_{\{E N G, U S\}}( } \Theta_{\{E N G, U S\}}\right)=1$ | $\begin{aligned} & m^{\Theta_{\{E N G, U S\}}(\{n o\})=0.05} \\ & m^{\Theta_{\{E N G, U S\}}}\left(\Theta_{\{E N G, U S\}}\right)= \\ & 0.95 \end{aligned}$ | $\begin{aligned} & m^{\Theta\{E N G, U S\}}(\{\text { yes }\})=0.4 \\ & m^{\Theta}{ }_{\{E N G, U S\}}\left(\Theta_{\{E N G, U S\}}\right)= \\ & 0.6 \end{aligned}$ |
| \{FR\} | $\begin{aligned} & m^{\Theta_{\{F R\}}}(\{\text { yes }\})=0.05 \\ & -m^{\Theta_{\{F R\}}}\left(\Theta_{\{F R\}}\right)=0.95 \end{aligned}$ | $\left.m^{\Theta_{\{F R\}}( } \Theta_{\{F R\}}\right)=1$ | $\begin{aligned} & m^{\Theta_{\{F R\}}}(\{\text { yes }\})=0.2 \\ & -m^{\Theta_{\{F R\}}}\left(\Theta_{\{F R\}}\right)=0.8 \end{aligned}$ |
| \{SW\} | $\begin{aligned} & m^{\Theta_{\{S W\}}(\{n o\})=0.4} \\ & -m^{\Theta_{\{S W\}}}\left(\Theta_{\{S W\}}\right)=0.6 \end{aligned}$ | $\begin{aligned} & m^{\Theta_{\{S W\}}(\{n o\})=0.2} \\ & -m^{\Theta_{\{S W\}}}\left(\Theta_{\{S W\}}\right)=0.8 \\ & \hline \end{aligned}$ | $m^{\Theta}{ }_{\{S W\}}\left(\Theta_{\{S W\}}\right)=1$ |

Table 4.22: Belief pair-wise matrix regarding C8 criterion

| C8 | $\{E N G, U S\}$ | $\{F R\}$ | $\{S W\}$ |
| :---: | :---: | :---: | :---: |
| $\{E N G, U S\}$ | $m^{\Theta\{E N G, U S\}}\left(\Theta_{\{E N G, U S\}}\right)=1$ | $\begin{aligned} & m^{\Theta_{\{E N G, U S\}}(\{n o\})=0.05} \\ & m^{\Theta_{\{E N G, U S\}}}\left(\Theta_{\{E N G, U S\}}\right)= \\ & 0.95 \end{aligned}$ | $\begin{aligned} & m^{\Theta_{\{E N G, U S\}}(\{y e s\})=0.4} \\ & \left.m^{\Theta_{\{E N G, U S\}}( } \Theta_{\{E N G, U S\}}\right)= \\ & 0.6 \end{aligned}$ |
| \{FR\} | $\begin{aligned} & m^{\Theta_{\{F R\}}(\{y e s\})}=0.05 \\ & -m^{\Theta_{\{F R\}}}\left(\Theta_{\{F R\}}\right)=0.95 \end{aligned}$ | $\left.m^{\Theta_{\{F R\}}( } \Theta_{\{F R\}}\right)=1$ | $\begin{aligned} & m^{\Theta_{\{F R\}}}(\{\text { yes }\})=0.2 \\ & -m^{\Theta_{\{F R\}}}\left(\Theta_{\{F R\}}\right)=0.8 \end{aligned}$ |
| \{SW\} | $\begin{aligned} & m^{\Theta_{\{S W\}}(\{n o\})=0.4} \\ & -m^{\Theta_{\{S W\}}}\left(\Theta_{\{S W\}}\right)=0.6 \\ & \hline \end{aligned}$ | $\begin{aligned} & m^{\Theta_{\{S W\}}(\{n o\})=0.2} \\ & -m^{\Theta_{\{S W\}}}\left(\Theta_{\{S W\}}\right)=0.8 \end{aligned}$ | $m^{\Theta}{ }^{\text {SSW }}\left(\Theta_{\{S W\}}\right)=1$ |

After the elicitation of the preferences of the decision maker and the identification of the belief pair-wise comparisons matrices, we can now use our implemented software to get the final rank of the identified alternatives. In fact, the main procedure of our extended belief AHP is then:

1. Computing the alternatives priorities using the belief pair-wise comparison approach introduced in the previous chapter. As a result, we

Table 4.23: Belief pair-wise matrix regarding C9 criterion

| C9 | $\{E N G, U S\}$ | $\{F R\}$ | $\{S W\}$ |
| :---: | :---: | :---: | :---: |
| $\{E N G, U S\}$ | $\left.m^{\Theta_{\{E N G, U S\}}( } \Theta_{\{E N G, U S\}}\right)=1$ | $\begin{aligned} & m^{\Theta_{\{E N G, U S\}}(\{n o\})=0.05} \\ & m^{\Theta_{\{E N G, U S\}}}\left(\Theta_{\{E N G, U S\}}\right)= \\ & 0.95 \end{aligned}$ | $\left.\begin{array}{l} m^{\Theta\{E N G, U S\}}(\{y e s\})=0.4 \\ m^{\Theta}(\{E N G, U S\} \\ 0.6 \end{array} \Theta_{\{E N G, U S\}}\right)=$ |
| \{FR\} | $\begin{aligned} & m^{\Theta_{\{F R\}}(\{y e s\})}=0.05 \\ & -m^{\Theta_{\{F R\}}}\left(\Theta_{\{F R\}}\right)=0.95 \end{aligned}$ | $\left.m^{\Theta_{\{F R\}}( } \Theta_{\{F R\}}\right)=1$ | $\begin{aligned} & m^{\Theta_{\{F R\}}(\{\text { yes }\})}=0.2 \\ & -m^{\Theta_{\{F R\}}}\left(\Theta_{\{F R\}}\right)=0.8 \end{aligned}$ |
| \{SW\} | $\begin{aligned} & m^{\Theta_{\{S W\}}(\{n o\})=0.4} \\ & -m^{\Theta_{\{S W\}}}\left(\Theta_{\{S W\}}\right)=0.6 \\ & \hline \end{aligned}$ | $\begin{aligned} & m^{\Theta_{\{S W\}}(\{n o\})=0.2} \\ & -m^{\Theta_{\{S W\}}}\left(\Theta_{\{S W\}}\right)=0.8 \end{aligned}$ | $m^{\left.\Theta_{\{S W}\right\}}\left(\Theta_{\{S W\}}\right)=1$ |

Table 4.24: Belief pair-wise matrix regarding C10 criterion

| C10 | $\{E N G, U S\}$ | $\{F R\}$ | $\{S W\}$ |
| :---: | :---: | :---: | :---: |
| $\{E N G, U S\}$ | $m^{\Theta}{ }_{\{E N G, U S\}}\left(\Theta_{\{E N G, U S\}}\right)=1$ | $\begin{aligned} & m^{\Theta_{\{E N G, U S\}}(\{n o\})=0.05} \\ & m^{\Theta_{\{E N G, U S\}}}\left(\Theta_{\{E N G, U S\}}\right)= \\ & 0.95 \end{aligned}$ | $\begin{aligned} & m^{\Theta_{\{E N G, U S\}}(\{y e s\})}=0.4 \\ & m^{\Theta_{\{E N G, U S\}}}\left(\Theta_{\{E N G, U S\}}\right)= \\ & 0.6 \end{aligned}$ |
| \{FR\} | $\begin{aligned} & m^{\Theta_{\{F R\}}(\{y e s\})}=0.05 \\ & -m^{\Theta_{\{F R\}}}\left(\Theta_{\{F R\}}\right)=0.95 \end{aligned}$ | $\left.m^{\Theta_{\{F R\}}( } \Theta_{\{F R\}}\right)=1$ | $\begin{aligned} & m^{\Theta_{\{F R\}}(\{y e s\})}\left(\begin{array}{l} \text { y } \\ -m^{\Theta_{\{F R\}}}\left(\Theta_{\{F R\}}\right)=0.8 \end{array}\right. \end{aligned}$ |
| \{SW\} | $\begin{aligned} & m^{\Theta_{\{S W\}}(\{n o\})=0.4} \\ & -m^{\Theta_{\{S W\}}}\left(\Theta_{\{S W\}}\right)=0.6 \\ & \hline \end{aligned}$ | $\begin{aligned} & m^{\Theta_{\{S W\}}(\{n o\})=0.2} \\ & -m^{\Theta_{\{S W\}}}\left(\Theta_{\{S W\}}\right)=0.8 \\ & \hline \end{aligned}$ | $\left.m^{\Theta_{\{S W}( } \Theta_{\{S W\}}\right)=1$ |

get ten bba's relative to each pair-wise comparison matrix with respect to each criterion.
2. Updating the alternatives' priorities. First, the reliability measure of the criterion $\{C 5\}$ and $\{C 6\}$ is directly discounted with their corresponding bba's. Then, for the subsets of criteria $\{C 1, C 3, C 7\}$ and $\{C 2, C 4, C 8, C 9, C 10\}$, we have combined their corresponding bba's using the conjunctive rule. After that, the obtained bba is discounted by the measure of reliability relative to the specific group of criteria.
3. Combining the overall bba's to get a single representation by using the conjunctive rule.
4. Ranking of the alternatives according to the pignistic probability.

Then, by using our implemented software, we get this final result (see Table 4.25):

Consequently, by applying the extended belief AHP approach on our real application problem, the decision maker is recommended to choose the alternative $\{F R\}$ since it has the highest pignistic probabilities 0.322.

In this application problem, both belief AHP approach and extended belief AHP yield to the same alternative ranking.

Table 4.25: The final result with belief pair-wise comparison

|  | Belief AHP |
| :---: | :---: |
| ENG | 0.23 |
| FR | 0.322 |
| USA | 0.178 |
| SW | 0.27 |

However, from this study it cannot be concluded that one method is more preferable compared to another, but, each method is applied in a particular context. So, the decision maker has the choice to select the method that best fit with the real problem situation.

### 4.4 Conclusion

In this chapter, the major variables and the main programs that we have used in order to implement our belief MCDM method are detailed. The principal belief AHP algorithm is also presented.

Then, we have shown the flexibility and feasibility of our proposed approaches by applying it on a real application problem. On one hand, in this study the belief AHP method is applied because it offers a process of judgment which allows the reduction of the number of pair-wise comparisons. On the other hand, the extended belief AHP approach is also applied on our application to solve the problem of uncertainty that may appear in expressing the preferences of the expert.

## Conclusion

In real-life decision making situation, the decision maker may encounter several difficulties when expressing his own level of preferences between alternatives or also criteria. However, standard version of AHP method is badly adapted to ensure its role in such environment. Thus, the need of the development of appropriate approach to this kind of environment is vital.

In this master thesis, we have defined a new AHP approach appropriate to this kind of environment. For this purpose, we were interested in belief function theory which presents an appropriate framework to handle the uncertainty related to the elicitation of preferences of the decision maker. So, we have developed what we call belief AHP method, a combination between the AHP method and the belief function theory.

The uncertainty that our approach is dealing with concerns two levels namely the criteria and the alternatives. The proposed method allows the decision maker to express his preferences with some degrees of uncertainty. In fact, our proposed approach allows the decision maker to express the importance of criteria with incomplete and imprecise preferences. So, the decision maker determines his opinions on groups of criteria instead of single one. To rank the alternatives, our method is able to use sets of criteria to compare sets of alternatives, which can help the decision maker to express subjective judgments between these alternatives.

Another part of the work provides some criticisms according to the comparison procedure. In fact, we have extended the belief AHP approach on a more flexible method by introducing uncertainty in the pair-wise comparison matrix. Thus, to evaluate the responses of the pair-wise comparison question, the decision maker expresses his judgment with some degrees of uncertainty. This uncertainty is represented via basic belief assignments (bba's).

Then, we have applied the two proposed approach on a real application
problem: the life cycle assessment, to demonstrate an applicable way of improving evaluation tactics in complex decision problems, where there is a large number of criteria and there is a need to follow human behavior.

Nevertheless, the proposed work is still subject to improvements. It can be extended into different directions. In fact, our method can be improved by defining a new consistency ration in a belief function framework. Thus, the proposed method will be more flexible, if it will be able to calculate the consistency of a belief pair-wise matrix.

In addition to the uncertainty on elicitation of preferences, another line of research will be to assume that each set of criteria can be defined with a degree of uncertainty. This uncertainty will be represented by a bba.

An interesting future work is to make our method able to solve more complex hierarchical problems. That is the hierarchical structure is characterized by more than three levels: the overall objective, criteria, sub-criteria and alternatives.

Further work may be suggested on the connection between the belief AHP approach and other methods. It would be interesting to combine our proposed approach with the technique for order preference by similarity to ideal solution (TOPSIS) method, the simple additive weighting (SAW) method.

## Bibliography

Barnett, J. (1991). Calculating Dempster-Shafer plausibility. IEEE Transactions on Pattern Analysis and Machine Intelligence, 13, 599-602.
Barron, F., \& Barrett, B. (1996). The efficacy of SMARTER: Simple MultiAttribute Rating Technique Extended to Ranking. Acta Psychologica.
Basak, I. (1998). Probabilistic judgments specified partially in the Analytic Hierarchy Process. European Journal of Operational Research, 108, 153-164.
Belton, V., \& Stewart, T. J. (2003). Multiple Criteria Decision Analysis: an integrated approach. Kluwer Academic Publishers.
Beynon, M. (2002). DS/AHP method: A mathematical analysis, including an understanding of uncertainty. European Journal of Operational Research, 140, 148-164.
Beynon, M., Curry, B., \& Morgan, P. (2000). The Dempster-Shafer theory of evidence: An alternative approach to multicriteria decision modelling. OMEGA, 28(1), 37-50.
Beynon, M., Xu, D. C., \& Marshall, D. (2001). An expert system for Multi-Criteria Decision Making using Dempster Shafer theory. Expert Systems with Applications, 20, 357-367.
Brans, J., Vincke, P., \& Marechal, B. (1986). How to select and how to rank projects: The PROMOTEE method. European Journal of Operational Research, 24, 228-238.
Buyukyaz, M., \& Sucu, M. (2002). The Analytic Hierarchy and Analytic Network Processes. Hacettepe Journal of Mathematics and Statistics, 65-73.
Chung, S. H., Lee, A. H. L., \& Pearn, W. L. (2006). Analytic network process (ANP) approach for product mix planning in semiconductor fabricator. International Journal of Production Economics, 96, 15-36.
Colorni, A., Paruccini, M., \& Roy, B. (2001). Multiple Criteria Decision Aiding. The European Commission.
Dempster, A. (1967). Upper and lower probabilities induced by a multiple
valued mapping. Annals of Mathematical Statistics, 38, 325-339.
Dempster, A. (1968). A generalization of bayesian inference. Journal of the Royal Statistical Society, 30, 205-247.
Denoeux, T. (2006). The cautious rule of combination for belief functions and some extensions. Proceedings of FUSION'2006. Florence, Italy.
Dezert, J., Tacnet, J. M., Batton-Hubert, M., \& Smarandache, F. (2010). Multi-Criteria Decision Making based on DSmT-AHP. Workshop on the Theory of Belief Functions.
Dubois, D., \& Prade, H. (1986). An approach to computerized processing of uncertainty. Int. J. Gen Syst, 12(3), 193-226.
Figueira, J., Greco, S., \& Ehrgott, M. (2005). Multiple Criteria Decision Analysis: state of the art surveys (Vol. 4). Springers International Series in Operations Research and Management Science.
Figueira, J., Mousseau, V., \& Roy, B. (2005). Electre methods. International Series in Operations Research and Management Science.
Holder, R. D. (1990). Some comments on the Analytic Hierarchy Process. Journal of the Operational Research Society, 41(11), 1073-1076.
Holder, R. D. (1995). Some comments on saaty's AHP. Management Science, 41, 1091-1095.
Inagaki, T. (1991). Interdependence between safety-control policy and multiple sensor scheme via Dempster-Shafer theory. IEEE Transactions on Reliability, 40(2), 182-188.
Joaquin, P. (1990). Some comments on the analytic hierarchy process. The Journal of the Operational Research Society, 41(6), 1073-1076.
Keeney, R., \& Raiffa, H. (1976). Decisions with multiple objectives: Preferences and value tradeoffs. Cambridge University Press.
Kohlas, J., \& Monney, P. A. (1995). A mathematical theory of hints: An approach to Dempster-Shafer theory of evidence. Lecture Notes in Economics and Mathematical Systems 425, Springer-Verlag.
Kong, F., \& Liu, H. (2005). Applying Fuzzy Analytic Hierarchy Process to evaluate success factors of E-commerce. International Journal Of Information And Systems Sciences, 1, 406-412.
Laarhoven, P. V., \& Pedrycz, W. (1983). A fuzzy extension of Saaty's priority theory. Fuzzy Sets and Systems, 11, 199-227.
Lefevre, E., Colot, O., \& Vannoorenberghe, P. (2002). Belief function combination and conflict management. Information Fusion, 3(2), 149-162.
Linkov, I., Varghese, A., Jamil, S., Kiker, G., \& Bridges, T. (2004). MultiCriteria Decision Analysis: A framework for structuring remedial decisions at contaminated sites. Kluwer, 38, 15-54.
Lipovetsky, S., \& Tishler, A. (1999). Interval estimation of priorities in the AHP. European Journal of Operational Research, 114, 153-164.

Lootsma, F. A. (1997). Fuzzy logic for planning and decision-making. Kluwer Academic Publishers.
Mellouli, K. (1987). On the propagation of beliefs in networks using the Dempter-Shafer theory of evidence. PhD thesis, School of business, University of Kansas.
Moreno-Jimeneza, J. M., \& Vargas, L. G. (1993). A probabilistic study of preference structures in the Analytic Hierarchy Process with interval judgments. Mathematical and Computer Modelling, 17, 73-81.
Parsons, S. (1994). Some qualitative approaches to applying the DS theory. Information and Decision Technologies, 19, 321-337.
Ra, J. (1999). Chainwise paired comparisons. Decision Sciences, 30, 581-599.
Roy, B. (1996). Multicriteria methodology for decision aiding. Kluwer Academic Publishers.
Saaty, T. (1977). A scaling method for priorities in hierarchical structures. Journal of Mathematical Psychology, 15, 234-281.
Saaty, T. (1980). The Analytic Hierarchy Process. McGraw-Hill, New-York.
Saaty, T. (1990). An exposition of the AHP in reply to the paper Remarks on the Analytic Hierarchy Process. Management Science, 36(3), 426-447.
Saaty, T. (1996). Decision making with dependence and feedback: The Analytic Network Process. Pennsylvania: RWS Publications.
Saaty, T. (2003). Decision-making with the AHP: Why is the principal eigenvector necessary. European Journal of Operational Research, 145, 85-91.
Saaty, T. (2008). Decision making with the Analytic Hierarchy Process. Int. J. Services Sciences, 1, 83-98.

Saaty, T. L., \& Vargas, L. G. (2001). Models, methods, concepts and applications of the Analytic Hierarchy Process. Kluwer Academic Publishers.
Schoner, B., \& Wedley, W. C. (1989). Ambiguous criteria weights in AHP: consequences and solutions. Decision Sciences, 20, 462-475.
Shafer, G. (1976). A mathematical Theory of Evidence. Princeton University Press.
Shafer, G. (1986). The combination of evidence. International Journal of Intelligent Systems, 1, 155-180.
Shafer, G. (1990). Perspectives on the theory and practice of belief functions. International Journal of Approximate Reasoning, 323-362.
Shafer, G., \& Pearl, J. (1990). Readings in uncertain reasoning. Morgan Kaufmann.
Smarandache, F., \& Dezert, J. (2004). Advances and applications of DSmT for information fusion (Collected works). American Research Press.
Smets, P. (1990). The combination of evidence in the Transferable Belief Model. IEEE Pattern analysis and Machine Intelligence, 447-458.

Smets, P. (1991). The Transferable Belief Model and other interpretations of Dempster-Shafer's model. In (P. P. Bonissone and M. Henrion and L. N. Kanal and J. F. Lemmer ed., pp. 375-384). Uncertainty in Artificial Intelligence 6.
Smets, P. (1992). Transferable Belief Model for expert judgments and reliability problems. Reliability Engineering and System Safety, 38, 59-66.
Smets, P. (1993). Belief functions: The disjunctive rule of combination and the generalized bayesian theorem. International Journal of Approximate Reasoning, 9, 1-32.
Smets, P. (1995). The canonical decomposition of a weighted belief. In Proceedings of the Fourteenth International Joint Conference on Artificial Intelligence, 1896-1901.
Smets, P. (1998). The application of the Transferable Belief Model to diagnostic problems. International Journal of Intelligent Systems, 13, 127-158.
Smets, P., \& Kennes, R. (1994). The Transferable Belief Model. Artificial Intelligence, 66, 191-234.
Tang, Y., \& Beynon, M. J. (2005). Application and development of a Fuzzy Analytic Hierarchy Process within a capital investment study. International Journal Of Information And Systems Sciences, 1(2), 207-230.
Triantaphyllou, E. (2000). Multi-Criteria Decision Making methods: a comparative study. Kluwer Academic Publishers.
Utkin, L. V., \& Simanova, N. V. (2008). Multi-Criteria Decision Making by incomplete preferences. Journal of Uncertain Systems, 2, 255-266.
Yager, R. R. (1987). On the Dempster-Shafer framework and new combination rules. Information Sciences, 41, 93-137.
Yang, J. B., \& Singh, M. G. (1994). An Evidential Reasoning Approach for multiple attribute decision making with uncertainty. IEEE transactions on systems, man, and cybernetics, 24(1), 1-18.
Zeleny, M. (1982). Multiple Criteria Decision Making. McGraw-Hill Book Company.

