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A THEORETICAL FRAMEWORK FOR POSSIBILISTIC INDEPENDENCE IN A WEAKLY ORDERED SETTING

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The notion of independence is central in many information processing areas, such as multiple criteria decision making, databases, or uncertain reasoning. This is especially true in the later case, where the success of Bayesian networks is basically due to the graphical representation of independence they provide. This paper first studies qualitative independence relations when uncertainty is encoded by a complete pre-order between states of the world. While a lot of work has focused on the formulation of suitable definitions of independence in uncertainty theories our interest in this paper is rather to formulate a general definition of independence based on purely ordinal considerations, and that applies to all weakly ordered settings. The second part of the paper investigates the impact of the embedding of qualitative independence relations into the scale-based possibility theory. The absolute scale used in this setting enforces the commensurateness between local pre-orders (since they share the same scale). This leads to an easy decomposability property of the joint distributions into more elementary relations on the basis of the independence relations. Lastly we provide a comparative study between already known definitions of possibilistic independence and the ones proposed here.

Keywords: Knowledge representation, possibilistic independence, weakly ordered setting.

1. Introduction

Various notions of (in)dependence are central in multiple criteria analysis[34], in relational data decomposition[46], or in uncertain reasoning based on Bayesian networks[33,41] or logical reasoning[6,8,35,36]. There has been a considerable interest in artificial intelligence, in the last few years, for discussing independence in various representation frameworks, due to the success of Bayesian networks. Conditional independence relations between variables play an important role in the handling of uncertain information.

From an operational point of view, two forms of independence can be distinguished:
decompositional independence which ensures the decomposition of a joint distribution pertaining to tuples of variables into local distributions on smaller subsets of variables. The reasoning machinery can then work at the local level without losing any information.
causal independence for expressing the lack of causality between variables. This form of independence is always characterized in semantic terms. Roughly speaking, a variable (or set of variables) is said to have no influence on another variable (or set of variables) if our belief in the value of the latter does not change when learning something about the value of the former.

Contrary to decompositional independence, causal independence relations are not necessarily *symmetric*. In other words, if a variable A is independent of B, we are not sure that B is independent of A. These two kinds of independence relations are not necessarily mutually exclusive. Ideally, a good definition of independence expresses both the lack of causality (so it can be easily expressed by experts), and is useful for computations.

In the probabilistic framework, two variables A and B are said to be *decomposably* independent if the joint probability on the range of (A, B) is the product of the probability distribution of A and the probability distribution of B, i.e., $P(A \wedge B) = P(A) \cdot P(B)$. And A and B are said to be *causally* independent if the probability of B given A is the same as the probability of B, i.e., $P(B \mid A) = P(B)$. In this framework, it can be easily checked that causal and decompositional independence relations are equivalent for positive distributions.

In possibility theory, and more generally in weakly ordered settings, the situation is different since causal and decompositional relations are not always equivalent. In this paper we investigate possible definitions of independence in two settings, using qualitative plausibility relations, or possibility distributions ranging on the scale $[0, 1]$. Different works have been achieved on independence relations: de Campos and Huete[9,10], Fonck[24,25], Studený[44], de Cooman and Kerre[11], Vejnarová [45].
However, this paper differs from the previous ones since the proposed independence relations are only based on the qualitative plausibility relations induced by possibility distributions.

This paper is organized as follows: The first part proposes and investigates, independence relations in comparative possibility theory where only the plausibility relations underlying the possibility distributions are used.
Then, we study the independence in scale-based possibility theory. Lastly we provide a comparative study between existing possibilistic independence and proposed qualitative independence relations. Results of this article extend those of two conference papers[1,2]. The proofs of main results are given in the appendix.

2. Notations and definitions

We first give some notations and definitions used in this article.
Let $V = \{A_1, A_2, ..., A_N\}$ be a set of variables. We denote by D_A the supposedly finite domain associated with the variable A. By a_i we denote any instance of A_i. $X, Y, Z, ...$ denote subsets of variables from V, and $D_X = \times_{A_i \in X} D_{A_i}$ represents the Cartesian product of domains of variables in X. By x we denote any instance of X; if $X = \{A_1, ..., A_n\}$ then $x = (a_1, ..., a_n)$. $\Omega = \times_{A_i \in V} D_{A_i}$ denotes the universe of discourse, which is the Cartesian product of all variable domains in V. Each element $\omega \in \Omega$ is called a possible world or state of Ω. Depending on the context, we use one of the following notations: either tuples: $\omega = (a_1, ..., a_N)$ or conjunctions: $\omega = a_1 \wedge ... \wedge a_N$, then $\omega[A_i] = a_i$.
ϕ, ψ, ξ denote the subclasses of Ω (called propositions or events) and $\neg\phi$ denotes the complementary set of ϕ i.e. $\neg\phi = \Omega - \phi$. $\phi \wedge \psi$ (resp. $\phi \vee \psi$) denotes the intersection (resp. the union) of ϕ and ψ. Similarly, ϕ_X denotes a subset of D_X.
$[a_i] = \{\omega = (a_1 \wedge ... \wedge a_N) : A_i = a_i\}$ denotes the set of states whose i^{th} component is a_i. Similarly, $\forall x \in D_X, [x] = \{\omega = (a_1 \wedge ... \wedge a_n) : \forall A_i \in X = \{A_1, ..., A_n\}, A_i = a_i\}$ denotes the set of states whose restrictions to variables in X is x.

When there is no ambiguity, we use x instead of $[x]$ and $x \wedge y$ (resp. $x \vee y$) instead of $[x] \wedge [y]$ (resp. $[x] \vee [y]$).

In the rest of this paper, we will often refer to the following example to illustrate different notions of independence:

Example 1 *Suppose that in a cultivated field, we have information about the physiological accidents that can affect the culture due to bacteria, mushrooms etc., the maintenance (chemical fertilizers, etc.) and the land yield, then:*

- *We can distinguish three variables i.e., physiological accidents (Pacc), maintenance (Maint) and land yield (Yield) thus* $V = \{PAcc, Maint, Yield\}$,
- *The domains associated with these variables are :*
  $D_{PAcc} = \{Disease1(d1), Disease2(d2), NoDisease(nd)\}$,
  $D_{Maint} = \{Good(gm), Medium(mm), Weak(wm)\}$,
  $D_{Yield} = \{Good(gy), Weak(wy)\}$

  Note that for the sake of simplicity, in some examples we only use binary variables. This will be made precise in each use.
- *The set of all states is* $\Omega = D_{PAcc} \times D_{Maint} \times D_{Yield}$.
- *A possible state is that there is no disease, and that the maintenance and the yield are good:* $\omega = nd \wedge gm \wedge gy$. *Then* $\omega[PAcc] = nd, \omega[Maint] = gm$ *and* $\omega[Yield] = gy$.
- *The set* $[nd] = \{nd \wedge gm \wedge gy, nd \wedge gm \wedge wy, nd \wedge mm \wedge gy, nd \wedge mm \wedge wy, nd \wedge wm \wedge gy, nd \wedge wm \wedge wy\}$ *denotes the set of states where the instance nd of the*

variable $PAcc$ holds.
The set $[gm] = \{d1 \wedge gm \wedge gy, d1 \wedge gm \wedge wy, d2 \wedge gm \wedge gy, d2 \wedge gm \wedge wy, nd \wedge gm \wedge gy, nd \wedge gm \wedge wy\}$ denotes the set of states where the instance gm of the variable $Maint$ holds.
The models of the event $nd \wedge gm$ are $[nd \wedge gm] = \{nd \wedge gm \wedge gy, nd \wedge gm \wedge wy\} = [nd] \cap [gm]$ where $\cap$ is the set intersection symbol.

3. Comparative possibility theory

In the following, we give a formal description of the qualitative representation of uncertainty we are using. The basic idea is to equip the referential Ω with a *complete pre-order*[1] also called a *weak order*, instead of using a totally ordered scale. This weak order denoted $\geq_\pi$, corresponds to a *plausibility relation* (also called a *comparative possibility relation*) on Ω and simply enables us to express that some situations are more plausible than others. We denote $=_\pi$ (resp. $>_\pi$, $<_\pi$) the equality (resp. inequality) relation corresponding to $\geq_\pi$. Namely the relation $\omega =_\pi \omega'$ means that ω is as plausible as ω'. We now give some definitions regarding to plausibility relations:

- *Most plausible states*: Given $\xi = \{\omega_1, .., \omega_n\} \subseteq \Omega$, the most plausible state(s) in the set ξ is defined by $max(\xi)$ s.t.

$$max(\xi) = \{\omega_i : \omega_i \in \xi, \nexists \omega_j \in \xi \text{ s.t. } \omega_j >_\pi \omega_i\}. \tag{1}$$

- *Least plausible states*: Given $\xi = \{\omega_1, .., \omega_n\} \subseteq \Omega$, the least plausible state(s) in the set ξ is defined by $min(\xi)$ s.t.

$$min(\xi) = \{\omega_i : \omega_i \in \xi, \nexists \omega_j \in \xi \text{ s.t. } \omega_i >_\pi \omega_j\}. \tag{2}$$

- Given a relation $\geq_\pi$ on Ω, we can lift it to another plausibility relation defined on the subsets of Ω denoted $\geq_\Pi$ by (e.g.,[12]):

$$\phi \geq_\Pi \psi \text{ iff } \forall \omega \in \psi, \exists \omega' \in \phi \text{ such that } \omega' \geq_\pi \omega. \tag{3}$$

  Namely, $\phi \geq_\Pi \psi$ holds if a best element in ϕ is preferred to best element(s) in ψ. In other terms:

$$\phi \geq_\Pi \psi \text{ iff } \exists \omega \in max(\phi), \omega' \in max(\psi) \text{ such that } \omega \geq_\pi \omega'.$$

  The idea behind the relation $\geq_\Pi$ is that the agent whose epistemic state is modeled by the plausibility relation $\geq_\pi$ evaluates events by their most plausible state considering that if ϕ occurs, then the expected situation is among the states in $max(\phi)$ because they are normal states.

[1] A relation $\geq$ on Ω is a complete pre-order if $\geq$ is reflexive, transitive and for all ω_1, ω_2, we have either $\omega_1 \geq \omega_2$ or $\omega_2 \geq \omega_1$.

This qualitative representation of uncertainty is also used in several non-monotonic formalisms like Lehmann's ranked models[37], plausibility relations[31], possibility theory, Spohn's ordinal conditional functions[43,42] and system of spheres[30].

In particular Spohn[43,42] represents plausibility orderings by means of well-ordered partitions $\{\phi_1, ..., \phi_p\}$ such that:

$$\forall i \in \{1, .., p\}, \forall \omega, \omega' \in \phi_i : \omega =_\pi \omega',$$

$$\forall i < j \text{ s.t. } i \in \{1, .., p\}, j \in \{1, .., p\}, \forall \omega \in \phi_i, \forall \omega' \in \phi_j : \omega >_\pi \omega',$$

that is $\phi_1 = max(\Omega), \phi_p = min(\Omega)$. Thus, ϕ_1 contains the most plausible states of the world. When $\phi_1 = \Omega$, the plausibility relation $\geq_\pi$ is *uniform* and expresses complete ignorance.

For any subset $X \subseteq V$, the projection of $\geq_\pi$ on D_X is denoted by $\geq_\pi^X$ and is defined by:

$$x \geq_\pi^X x' \text{ iff } [x] \geq_\Pi [x']. \tag{4}$$

If the projection of $\geq_\pi$ on D_X is uniform, then the agent is ignorant about the subset of variables X, or in other words, X is not informed, otherwise there is a proper subset $\phi_X^* \subseteq D_X$ of plausible values of X, such that $\phi_X^* = max(D_X)$.

The comparative possibility relations satisfies the characteristic property[12]:

$$\phi \geq_\Pi \psi \Rightarrow \phi \vee \xi \geq_\Pi \psi \vee \xi.$$

The dual necessity relation is defined by:

$$\phi \geq_N \psi \text{ iff } \neg\psi \geq_\Pi \neg\phi \text{ iff } max(\neg\psi) \geq_\Pi max(\neg\phi). \tag{5}$$

$\phi \geq_N \psi$ means that the agent is more certain about ϕ than about ψ. This relation is closely related to the one of epistemic entrenchment[29,21].

3.1. *Qualitative conditioning*

Conditioning is a crucial notion when studying independence relations. In the comparative setting, it consists in focusing a plausibility relation $\geq_\pi$ on a subclass $\phi \subseteq \Omega$, on the basis of a new piece of sure information about a case at hand; a plausibility relation restricted to ϕ, denoted by $\geq_{\pi|\phi}$ is obtained for answering questions on the case at hand for which only ϕ is known. We denote $=_{\pi|\phi}$ (resp. $>_{\pi|\phi}$, $<_{\pi|\phi}$) the equality (resp. inequality) relation corresponding to $\geq_{\pi|\phi}$. Natural postulates for qualitative conditioning are:

A$_1$**:** $\forall \omega_1, \omega_2 \in \phi, \omega_1 >_\pi \omega_2$ iff $\omega_1 >_{\pi|\phi} \omega_2$,

A$_2$**:** $\forall \omega_1 \in \phi, \forall \omega_2 \notin \phi, \omega_1 >_{\pi|\phi} \omega_2$,

A$_3$**:** $\forall \omega_1, \omega_2 \notin \phi, \omega_1 =_{\pi|\phi} \omega_2$.

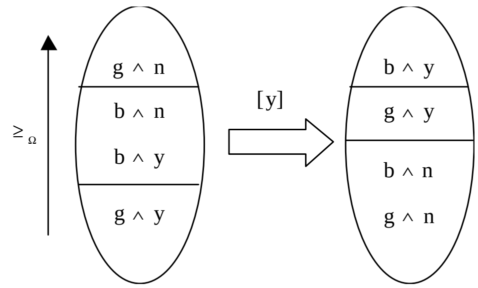

Figure 1: Qualitative conditioning

$\mathbf{A}_1$ means that the new plausibility relation should not alter the initial order between the elements of ϕ. $\mathbf{A}_2$ confirms that each model of ϕ should be preferred to any model not belonging to ϕ. Finally, the last postulate $\mathbf{A}_3$ says that the elements not belonging to ϕ are irrelevant and should be in the same equivalence class.
These three postulates determine in a **unique** manner the new plausibility relation $\geq_{\pi|\phi}$. Indeed, $\leq_{\pi|\phi}$ is obtained from $\leq_{\pi}$ by preserving the relative ordering between elements of ϕ, forcing elements which are outside ϕ to be equally plausible, but less plausible than any element of ϕ. This construction is illustrated by the following example.

Example 2 *Let us consider two binary variables, relative to climatic conditions (CCdt) and physiological accidents (PAcc), such that* $D_{CCdt} = \{Good(g), Bad(b)\}$, $D_{PAcc} = \{Yes(y), No(n)\}$ *with the following plausibility relation:*

$$g \wedge n >_{\pi} b \wedge y =_{\pi} b \wedge n >_{\pi} g \wedge y.$$

Consider, now that we receive a sure piece of information indicating that there is an accident ($[y] = \{b \wedge y, g \wedge y\}$), then the initial plausibility relation will be modified into the following, unique, relation (see Figure 1): $b \wedge y >_{\pi|\phi} g \wedge y >_{\pi|\phi} g \wedge n =_{\pi|\phi} b \wedge n$. *Indeed, from* $\mathbf{A}_2$*, we have* $b \wedge y >_{\pi|\phi} g \wedge n$, $b \wedge y >_{\pi|\phi} b \wedge n$, $g \wedge y >_{\pi|\phi} g \wedge n$ *and* $g \wedge y >_{\pi|\phi} b \wedge n$. *Moreover, from* $\mathbf{A}_3$*, we have* $b \wedge n =_{\pi|\phi} g \wedge n$. *Then, from* $\mathbf{A}_1$*, we have* $b \wedge y >_{\pi|\phi} g \wedge y$.

The conditional possibility ordering $\geq_{\pi|\phi}$ induces a conditional possibility ordering $\geq_{\Pi|\phi}$ between events simply defined as follows:

$$\alpha \geq_{\Pi|\phi} \beta \text{ iff } \alpha \wedge \phi \geq_{\Pi} \beta \wedge \phi.$$

Note that this kind of conditioning completely ignores the previous order between elements outside ϕ. Viewed as a revision process, conditioning imposes that all states in $\neg\phi$ become impossible, because ϕ is learned to be absolutely true. This is different in what is usually used in belief revision[29]. Indeed, for instance natural belief revision[7,43,42], considers minimal change for taking ϕ into account. It simply consists in moving the best elements in ϕ to the top level, and leaving the

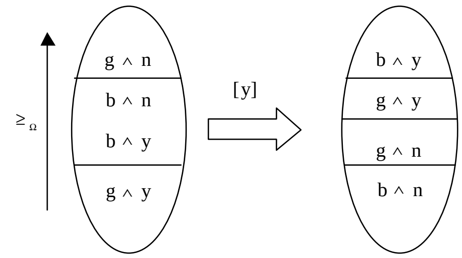

Figure 2: Natural belief revision

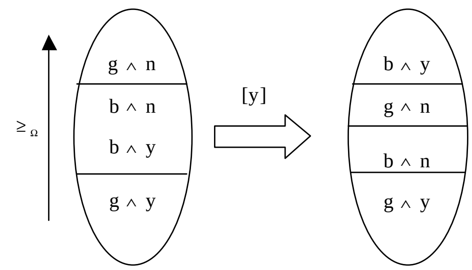

Figure 3: Revision in Papini's approach

order between other states unchanged. In our example, Figure 2 illustrates natural belief revision. Another example of belief revision is Papini's approach[40] which is obtained from A_1, A_2 and the following postulate:
$\mathbf{A}_4$: $\forall \omega_1, \omega_2 \notin \phi, \omega_1 >_\pi \omega_2$ iff $\omega_1 >_{\pi|\phi} \omega_2$.
In our example, this revision mode corresponds to Figure 3.

3.2. *Accepted beliefs*

We now introduce the notion of accepted belief which will be helpful for defining qualitative independence in Section 5, e.g.[22,28].

A proposition (or an event) ϕ is said to be accepted by the agent with epistemic state $\geq_\pi$ if and only if $\phi >_N \neg\phi$[16]. In particular, the set $\{\phi \text{ s.t. } \phi >_N \neg\phi\}$ is deductively closed. In other words, the subclasses of Ω are shared into three families: accepted beliefs ϕ such that $\phi >_\Pi \neg\phi$, rejected beliefs ϕ such that $\neg\phi >_\Pi \phi$ and ignored beliefs ϕ such that $\phi =_\Pi \neg\phi$. This trichotomy can be encoded as follows:

Definition 1 *The acceptance function associated with a plausibility relation* $\geq_\pi$ *denoted by* $\mathbf{Acc}_{\geq_\pi}(.)$ *assigns to each* ϕ *a value in* $\{-1, 0, 1\}$ *in the following way:*

$$\mathbf{Acc}_{\geq_\pi}(\phi) = \begin{cases} 1 & if\ \phi >_\Pi \neg\phi \\ -1 & if\ \neg\phi >_\Pi \phi \\ 0 & if\ \phi =_\Pi \neg\phi. \end{cases} \tag{6}$$

When $\mathbf{Acc}_{\geq_\pi}(\phi) = 1$ (resp. $\mathbf{Acc}_{\geq_\pi}(\phi) = -1$) we say that ϕ is accepted (resp.

rejected). $\mathbf{Acc}_{\geq_\pi}(\phi) = \mathbf{Acc}_{\geq_\pi}(\neg\phi) = 0$, corresponds to the situation of total ignorance concerning ϕ, i.e., ϕ and $\neg\phi$ are equally plausible.

Lemma 1 *The acceptance function is equivalently defined as follows:*

$$Acc_{\geq_\pi}(\phi) = \begin{cases} 1 & \text{if } max(\Omega) \subseteq \phi \\ -1 & \text{if } max(\Omega) \subseteq \neg\phi \\ 0 & otherwise. \end{cases} \quad (7)$$

Proposition 1 *The properties of the acceptance function* $\mathbf{Acc}_{\geq_\pi}$ *are as follows:*

1. *It is* monotonic *i.e.* $\phi \subseteq \psi \Rightarrow \mathbf{Acc}_{\geq_\pi}(\phi) \leq \mathbf{Acc}_{\geq_\pi}(\psi)$
2. $\mathbf{Acc}_{\geq_\pi}(\phi \wedge \psi) = 1$ *iff* $\mathbf{Acc}_{\geq_\pi}(\phi) = 1$ *and* $\mathbf{Acc}_{\geq_\pi}(\psi) = 1$
   this is because the set of accepted propositions is deductively closed.
3. $\mathbf{Acc}_{\geq_\pi}(\phi \wedge \psi) = min(\mathbf{Acc}_{\geq_\pi}(\phi), \mathbf{Acc}_{\geq_\pi}(\psi))$ *except if*
   $\mathbf{Acc}_{\geq_\pi}(\phi \wedge \psi) = -1$ *and* $\mathbf{Acc}_{\geq_\pi}(\phi) = \mathbf{Acc}_{\geq_\pi}(\psi) = 0.$
4. $\mathbf{Acc}_{\geq_\pi}(\neg\phi) = -\mathbf{Acc}_{\geq_\pi}(\phi)$
5. $\mathbf{Acc}_{\geq_\pi}(\phi \vee \psi) = max(\mathbf{Acc}_{\geq_\pi}(\phi), \mathbf{Acc}_{\geq_\pi}(\psi))$ *except if*
   $\mathbf{Acc}_{\geq_\pi}(\phi \vee \psi) = 1$ *and* $\mathbf{Acc}_{\geq_\pi}(\phi) = \mathbf{Acc}_{\geq_\pi}(\psi) = 0.$

Property 3 of this proposition is proved in the appendix. Properties 1, 2 and 4 are obvious consequences of Lemma 1 and property 5 is trivial using properties 3 and 4.

Property 2 confirms that the logic of accepted unconditional events is *classical* logic. Properties 3 and 5 point out an almost compositionality of $\mathbf{Acc}_{\geq_\pi}$ but, it is not compositional, and $\mathbf{Acc}_{\geq_\pi}$ should not be confused with a three-valued truth function.

The function $\mathbf{Acc}_{\geq_\pi}$ can be extended in order to take into account a given context. Then a conditional belief measure denoted by $\mathbf{Acc}_{\geq_\pi}(.|.)$ is defined by:

$$\mathbf{Acc}_{\geq_\pi}(\phi \mid \psi) = \begin{cases} 1 & if\ \phi \wedge \psi >_\Pi \neg\phi \wedge \psi \\ 0 & if\ \phi \wedge \psi =_\Pi \neg\phi \wedge \psi \\ -1 & if\ \neg\phi \wedge \psi >_\Pi \phi \wedge \psi. \end{cases} \quad (8)$$

When $\mathbf{Acc}$ is defined on subsets of Ω, we talk about *plain beliefs*, while when it is defined on conditionals we talk about *conditional beliefs*. In a fixed context ψ, $\mathbf{Acc}_{\geq_\pi}(. \mid \psi)$ enjoys the same properties as function $\mathbf{Acc}_{\geq_\pi}$.

Example 3 *Let us consider two binary variables A and B with the following plausibility relation* $a_1 \wedge b_1 >_\pi a_2 \wedge b_1 >_\pi a_1 \wedge b_2 =_\pi a_2 \wedge b_2$. *Then, for instance:*
$\mathbf{Acc}_{\geq_\pi}(a_1) = 1, \mathbf{Acc}_{\geq_\pi}(a_2) = -1,$
$\mathbf{Acc}_{\geq_\pi}(b_1) = 1, \mathbf{Acc}_{\geq_\pi}(b_2) = -1,$
$\mathbf{Acc}_{\geq_\pi}(a_1 \mid b_1) = 1, \mathbf{Acc}_{\geq_\pi}(a_1 \mid b_2) = 0,$
$\mathbf{Acc}_{\geq_\pi}(a_2 \mid b_1) = -1, \mathbf{Acc}_{\geq_\pi}(a_2 \mid b_2) = 0.$

Remarks:

- The plausibility relation $\geq_\pi$ determines $\mathbf{Acc}_{\geq_\pi}$ in a unique manner. The converse is not true. Namely, many plausibility relations can generate the same set of **plain beliefs**, i.e, we can have the same $\mathbf{Acc}_{\geq_\pi}$ on all events (including the states). Indeed, two plausibility relations induce the same plain beliefs if and only if they share the same set of most plausible states, as obviously stated by Lemma 1. The other parts of the relations may thus differ.

- Restricting the function $\mathbf{Acc}_{\geq_\pi}$ to Ω, we can distinguish three cases (we note $\mathbf{Acc}_{\geq_\pi}(\{\omega\}) = \mathbf{Acc}_{\geq_\pi}(\omega)$):

  - $\mathbf{Acc}_{\geq_\pi}(\omega) = 1$: in this case, ω is the unique state such that $\omega >_\pi \omega', \forall\omega' \neq \omega \in \Omega$. ω is then called the *accepted state* since $\{\omega\} >_N \{\omega'\}$ as well for any $\omega' \neq \omega$. Note that, if $\exists\omega$ such that, $\mathbf{Acc}_{\geq_\pi}(\omega) = 1$, then $\forall\omega' \neq \omega, \mathbf{Acc}_{\geq_\pi}(\omega') = -1$.
  - When $max(\Omega)$ contains more than one plausible instance then $\mathbf{Acc}_{\geq_\pi}(\omega) \leq 0, \forall\omega \in \Omega$. More precisely, $\forall\omega \in max(\Omega), \mathbf{Acc}_{\geq_\pi}(\omega) = 0$
  - $\mathbf{Acc}_{\geq_\pi}(\omega) = -1$ is equivalent to $\omega \notin max(\Omega)$, i.e. ω is not a plausible state.

  So, the function $\mathbf{Acc}_{\geq_\pi}(\omega)$ on *states* only distinguish between the most plausible states (i.e. $\mathbf{Acc}_{\geq_\pi}(x) \geq 0$) and the less plausible ones ($\mathbf{Acc}_{\geq_\pi}(x) = -1$). Interestingly, the restriction of $\mathbf{Acc}_{\geq_\pi}$ on Ω enables the function $\mathbf{Acc}_{\geq_\pi}$ to be reconstructed on all subsets of Ω.
  Indeed, $max(\Omega) = \{\omega \text{ s.t. } \mathbf{Acc}_{\geq_\pi}(\omega) = 1\} \cup \{\omega \text{ s.t. } \mathbf{Acc}_{\geq_\pi}(\omega) = 0\}$ (one of the sets is empty), and then it is enough to apply Lemma 1.

  So, $\mathbf{Acc}_{\geq_\pi}(\omega) = \mathbf{Acc}_{\geq'_\pi}(\omega), \forall\omega \in \Omega$, if and only if, $\mathbf{Acc}_{\geq_\pi}(\phi) = \mathbf{Acc}_{\geq'_\pi}(\phi), \forall\phi \subseteq \Omega$.

- However, the set of all *conditional beliefs* determines in a unique manner a plausibility relation on Ω. Indeed, the proof can be directly obtained by defining $\geq_\pi$ in the following way:

  $$\omega_1 >_\pi \omega_2 \text{ iff } \mathbf{Acc}_{\geq_\pi}(\{\omega_1\} \mid \{\omega_1, \omega_2\}) = 1 \tag{9}$$

  Example 4 *Let us consider the following conditional values $\mathbf{Acc}_\pi$ on some conditionals relative to the two binary variables A and B:*
  $\mathbf{Acc}_{\geq_\pi}(a_1 \mid b_1) = 1, \mathbf{Acc}_{\geq_\pi}(a_1 \mid b_2) = 0, \mathbf{Acc}_{\geq_\pi}(a_2 \mid b_1) = -1,$
  $\mathbf{Acc}_{\geq_\pi}(a_2 \mid b_2) = 0, \mathbf{Acc}_{\geq_\pi}(b_1 \mid a_1) = 1, \mathbf{Acc}_{\geq_\pi}(b_1 \mid a_2) = -1,$
  $\mathbf{Acc}_{\geq_\pi}(b_2 \mid a_1) = -1, \mathbf{Acc}_{\geq_\pi}(b_2 \mid a_2) = 1.$

  Using (8), the previous conditional beliefs induce, respectively:
  $a_1 \wedge b_1 >_\pi a_2 \wedge b_1$, $a_1 \wedge b_2 =_\pi a_2 \wedge b_2$, $a_1 \wedge b_1 >_\pi a_2 \wedge b_1$, $a_2 \wedge b_2 =_\pi a_1 \wedge b_2$,
  $a_1 \wedge b_1 >_\pi a_1 \wedge b_2$, $a_2 \wedge b_2 >_\pi a_2 \wedge b_1$, $a_1 \wedge b_1 >_\pi a_1 \wedge b_2$, $a_2 \wedge b_2 >_\pi a_2 \wedge b_1$.

We can check that these relations induce the following, unique, plausibility relation: $a_1 \wedge b_1 >_\pi a_1 \wedge b_2 =_\pi a_2 \wedge b_2 >_\pi a_2 \wedge b_1$.

Proposition 2 *The acceptance functions* $\mathbf{Acc}_{\geq_\pi}(.)$ *and the conditional acceptance function* $\mathbf{Acc}_{\geq_\pi}(.|.)$ *are related via the following Bayesian-like equation:*

$$\mathbf{Acc}_{\geq_\pi}(\phi \wedge \psi) = min(\mathbf{Acc}_{\geq_\pi}(\phi \mid \psi), \mathbf{Acc}_{\geq_\pi}(\psi)) \tag{10}$$

It may happen that $\mathbf{Acc}_{\geq_\pi}(\psi) = 1$ but $\mathbf{Acc}_{\geq_\pi}(\phi \wedge \psi) = 0$ or 1 or -1.

In the following, we use $\mathbf{Acc}(.)$ (resp. $\mathbf{Acc}(.|.)$) instead of $\mathbf{Acc}_{\geq_\pi}(.)$ (resp. $\mathbf{Acc}_{\geq_\pi}(.|.)$) when there is no ambiguity.

4. Scale-based possibility theory

In scale-based settings, uncertainty is handled in a qualitative way, but it is encoded on some linearly ordered scale (finite or infinite). Typical examples of these frameworks are possibility theory[20,23,48] where uncertainty is represented in the interval $[0, 1]$ and Spohn's ordinal functions[43,42] which use the set of integers. In the following, we only focus on possibility theory, but results of this article are also valid for other frameworks such as Spohn's ordinal functions, or Lehmann's ranked models[37], due to their close relation to possibility theory.

4.1. *Possibility distribution and possibility measure*

The basic concept in possibility theory is the notion of *possibility distribution*. It is a mapping from Ω to the scale [0, 1] usually denoted by π. Possibility distributions aim at encoding an agent's knowledge about an ill-known world : $\pi(\omega) = 1$ means that ω is completely possible and $\pi(\omega) = 0$ means that ω can not be the real world. A possibility distribution π is said to be *normalized* if there exists at least one state ω which is totally possible.

Given a possibility distribution π, the uncertainty of any event $\phi \subseteq \Omega$ is estimated by means of two dual measures:

- The **possibility measure** of ϕ:

$$\Pi(\phi) = \max_{\omega \in \phi} \pi(\omega). \tag{11}$$

  The measure $\Pi(\phi)$ evaluates at which level ϕ is **consistent** with our knowledge represented by the possibility distribution π.

- The **necessity measure**, associated with Π by duality:

$$N(\phi) = 1 - \Pi(\neg\phi) = \min_{\omega \notin \phi}(1 - \pi(\omega)). \tag{12}$$

  The measure $N(\phi)$ corresponds to the extent to which $\neg\phi$ is impossible and thus evaluates at which level ϕ is **certainly** implied by our knowledge (represented by the possibility distribution π).

Denoting $\mathbf{Conf}(\psi) = \Pi(\psi) - \Pi(\neg\psi)$, it is easy to see that the **Acc** function induced from π is such that $\mathbf{Acc}(\psi) = 1$ if $\mathbf{Conf}(\psi) > 0$, $\mathbf{Acc}(\psi) = -1$ if $\mathbf{Conf}(\psi) < 0$ and $\mathbf{Acc}(\psi) = 0$ if $\mathbf{Conf}(\psi) = 0$. $\mathbf{Conf}(\psi)$ corresponds to the idea of a confidence factor used in the expert systems literature.

4.2. *Possibilistic conditioning*

In the possibilistic setting conditioning consists in focusing our initial knowledge, encoded by a possibility distribution π on a subclass $\phi \subseteq \Omega$ due to the arrival of a new *certain* piece of evidence. The initial distribution π is then replaced by another one denoted by $\pi^{'} = \pi(. \mid \phi)$. We assume that $\phi \neq \emptyset$ and that $\Pi(\phi) > 0$. Natural postulates for possibilistic conditioning are:

$\mathbf{C}_1$: if $\pi(\omega) = 0$ then $\pi^{'}(\omega) = 0$,

$\mathbf{C}_2$: $\forall \omega \notin \phi, \pi^{'}(\omega) = 0$,

$\mathbf{C}_3$: $\pi^{'}$ should be normalized,

$\mathbf{C}_4$: $\forall \omega_1, \omega_2 \in \phi, \pi(\omega_1) > \pi(\omega_2)$ iff $\pi^{'}(\omega_1) > \pi^{'}(\omega_2)$

$\mathbf{C}_5$: if $\Pi(\phi) = 1$, then $\forall \omega \in \phi, \pi^{'}(\omega) = \pi(\omega)$.

$\mathbf{C}_1$ says that irrelevant states remain irrelevant after conditioning, $\mathbf{C}_2$ confirms that ϕ is a sure piece of information and $\mathbf{C}_3$ says that the result should be a normalized possibility distribution. Moreover, $\mathbf{C}_4$ says that the new possibility distribution should not affect the possibility degrees relative to the states in ϕ. Lastly, $\mathbf{C}_5$ says that if ϕ is already consistent with the beliefs encoded by π, then the possibility distribution remains unchanged on the models of ϕ. This is in agreement with the min-based combination mode which prevails in possibility theory; no further normalization is needed since $\Pi(\phi) = 1$.

Contrary to the comparative framework, the postulates ($\mathbf{C}_1$-$\mathbf{C}_5$) do not guarantee a *unique definition of conditioning*. Minimizing change leads to preserving the possibility degrees of elements in ϕ and assigning the degree 0 to others:

$$\pi(\omega \mid_m \phi) = \begin{cases} \pi(\omega) & \text{if } \omega \in \phi \\ 0 & \textit{otherwise}. \end{cases} \tag{13}$$

This method may obviously result in subnormal possibility distributions. Restoring the normalization, in order to satisfy $\mathbf{C}_3$, can be done in two different ways (when $\Pi(\phi) > 0$) depending on whether we are in a comparative setting where the scale $[0, 1]$ is only used for encoding an ordering between degrees (which may form a finite set of values), or if we are in a genuine numerical setting[20]:

- In an ordinal setting, we assign to the best elements of ϕ, the maximal possibility degree (i.e. 1), then we obtain:

$$\pi(\omega \mid_m \phi) = \begin{cases} 1 & if\ \pi(\omega) = \Pi(\phi) \text{ and } \omega \in \phi \\ \pi(\omega) & if\ \pi(\omega) < \Pi(\phi) \text{ and } \omega \in \phi \\ 0 & \textit{otherwise}. \end{cases} \tag{14}$$

This corresponds to the *min-based* conditioning.

- In a numerical setting, we proportionally shift up all elements of ϕ:

$$\pi(\omega \mid_p \phi) = \begin{cases} \frac{\pi(\omega)}{\Pi(\phi)} & \text{if } \omega \in \phi \\ 0 & \textit{otherwise.} \end{cases} \tag{15}$$

This corresponds to the *product-based* conditioning.

If $\Pi(\phi) = 0$ then, by convention $\pi(\omega \mid_m \phi) = \pi(\omega \mid_p \phi) = 1$.

Each of these two definitions of conditioning satisfies an equation close to the Bayesian rule, of the form:

$$\forall \omega,\ \pi(\omega) = \pi(\omega \mid \phi) \otimes \Pi(\phi) \tag{16}$$

respectively for $\otimes$ are the **min** (for (14)) and the **product** (for (15)) operators. The min-based conditioning (14) corresponds to the least specific solution of Equation 16 first proposed by Hisdal[32].

4.3. *Possibilistic framework vs Comparative framework*

Each possibility distribution π generates a unique plausibility relation $\geq_\pi$ defined by:

$$\omega \geq_\pi \omega' \text{ iff } \pi(\omega) \geq \pi(\omega'). \tag{17}$$

However, a plausibility relation corresponds to an infinity of possibility distributions on $[0, 1]$. It is natural to encode a weak order on Ω representing a plausibility relation $\geq_\pi$ and corresponding to a well-ordered partition $\{\phi_1, ..., \phi_p\}$, into a possibility distribution π ranging on a finite, totally ordered scale, with p levels.

Note that if $\geq_\pi$ is defined from π using (17) then:

$$\phi \geq_\Pi \psi \text{ iff } \Pi(\phi) \geq \Pi(\psi).$$

We now focus on the major differences between scaled possibility theory and the comparative framework. There are at least three differences between using possibility distributions or plausibility relations:

- *Normalization*: in possibility theory, fully plausible states receive the grade 1 ($\exists \omega \in \Omega$ s.t. $\pi(\omega) = 1$) while there is no counterpart in the comparative setting.
- *Existence of impossible states* graded to 0 in possibility theory, while all states are somewhat possible in the comparative setting.
- *Commensurability* between uncertainty levels when merging several possibility distributions, since all rankings reflect grades in the same scale $[0, 1]$.

Note that these remarks are also true for any scale-based representation framework and not only in possibility theory.

The normalization and the existence of impossible states explain why there are several definitions of possibilistic conditioning while there is a unique definition in the comparative setting. As we will show later, the commensurability property is crucial in the *decomposition* of some qualitative independence relations.

5. Independence in comparative possibility theory

In this section we propose several causal and decompositional definitions of possibilistic independence which apply to a plausibility relation $\geq_\pi$ on a Cartesian product of universes.

5.1. *Causal qualitative independence*

In the comparative setting, independence relations can be thought of either in terms of *qualitative plausibility relations* or in terms of *acceptance measures*. The two views can be related, as shown below where we present two possible definitions of causal independence. Basically, the variable set X is independent of Y if upon learning any instance of Y:
- the agent's beliefs on D_X i.e. the accepted (resp. rejected and ignored) instances of X are preserved, or
- the relative ordering between instances of X is preserved.

5.1.1. *Belief-preserving independence*

The first notion of causal independence in the ordinal setting is concerned with the preservation of accepted and rejected beliefs. A set of variables X can be considered as independent of Y in the context Z, if the accepted and rejected beliefs pertaining to X, held in the context Z, remain unchanged when some information about Y is obtained. Formally:

Definition 2 *Let $\geq_\pi$ be a plausibility relation defined on $\Omega = D_V$ and consider three mutually disjoint subsets of variables X, Y and Z forming a partition of V. X is said to be BP-independent (BP for Belief Preserving) of Y in the context Z, denoted by $I_{BP}(X, Z, Y)$, iff $\forall \phi_X \subseteq D_X, \forall \psi_Y \subseteq D_Y, \forall \xi_Z \subseteq D_Z$*

$$\mathbf{Acc}(\phi_X \mid \psi_Y \wedge \xi_Z) = \mathbf{Acc}(\phi_X \mid \xi_Z). \tag{18}$$

Compared with the notion of qualitative independence previously introduced by the authors[4,14,13] this definition is stronger in two extents: in[4,13] only particular events are concerned; moreover the idea was (especially in reference[13]) to preserve accepted beliefs only and not rejected ones.

Note that contrary to the situation in probability theory, BP-independence is not symmetric as shown by the counter example below.

Counter-example 1 *Consider two binary variables A and B with the following plausibility relation:* $a_1 \wedge b_1 >_\pi a_1 \wedge b_2 >_\pi a_2 \wedge b_2 >_\pi a_2 \wedge b_1$. *Table 1, shows that* $I_{BP}(A, \emptyset, B)$ *is true since* $\mathbf{Acc}(a \mid b) = \mathbf{Acc}(a), \forall a, b$. *However,* $I_{BP}(B, \emptyset, A)$ *is false, for instance* $\mathbf{Acc}(b_1) = 1 \neq \mathbf{Acc}(b_1 \mid a_2) = -1$.

Table 1: Lack of symmetry property for I_{BP}

a	b	$Acc(a \mid b)$	$Acc(a)$	$Acc(b \mid a)$	$Acc(b)$
a_1	b_1	1	1	1	1
a_1	b_2	1	1	-1	-1
a_2	b_1	-1	-1	-1	1
a_2	b_2	-1	-1	1	-1

It is then clear that $I_{BP}(X, Z, Y)$ means that fixing any instance z of Z, the set $\{x$ s.t. $x \wedge y \wedge z$ *is a plausible instance in* $D_X \wedge y \wedge z\}$ does not depend on y. Hence, knowing some information about Y does not alter accepted beliefs about X in context Z.

Definition 2 is stated for all events defined by X, Y and Z, respectively, since **Acc** is not a decomposable function. Nevertheless, it is enough to state it with instances of X, Y and Z only as stated by the following proposition.

Proposition 3 *Let $\geq_\pi$ be a plausibility relation defined on $\Omega = D_V$ and consider three mutually disjoint subsets of variables X, Y and Z forming a partition of V. The relation $I_{BP}(X, Z, Y)$ is true, iff,* $\forall x, y, z$,

$$\mathbf{Acc}(x \mid y \wedge z) = \mathbf{Acc}(x \mid z). \tag{19}$$

We denote by I_{BPS} the symmetrized version of BP-independence relation; i.e. the variable set X is said to be BPS-independent of Y in the context Z if:

$$(i)\ \mathbf{Acc}(x \mid y \wedge z) = \mathbf{Acc}(x \mid z) \text{ and}$$

$$(ii)\ \mathbf{Acc}(y \mid x \wedge z) = \mathbf{Acc}(y \mid z), \forall x, y, z. \tag{20}$$

BPS-independence relation preserves the *plausible* instances of X given Y and Y given X in context Z, but does not preserve the relative *ordering* between instances of X (resp. Y) in the context Y (resp. X) (except when restricting to binary variables).

Example 5 *Let A and B be two BPS-independent variables with the following plausibility relation $\geq_\pi$:* $a_1 \wedge b_1 >_\pi a_2 \wedge b_1 >_\pi a_3 \wedge b_1 >_\pi a_1 \wedge b_2 >_\pi a_2 \wedge b_2 =_\pi a_3 \wedge b_2$. *By projection, the local plausibility relation relative to A is then:*$a_1 >_\Pi a_2 >_\Pi a_3$.

However, in the context b_2, we have $a_1 >_\Pi a_2 =_\Pi a_3$, thus, the relative ordering between instances of A is not preserved in all contexts of B since $a_2 >_\Pi a_3$ while $a_2 =_\Pi a_3$ in the context b_2.

5.1.2. *Order-preserving independence*

The causality-oriented definition that we propose now simply says that X is independent of Y in the context of Z, if for all $z \in D_Z$, the local *preferential ordering* between the different instances of X is preserved after the revision by any instance y of Y. More formally:

Definition 3 *Let $\geq_\pi$ be a plausibility relation defined on $\Omega = D_V$ and consider three mutually disjoint subsets of variables X, Y and Z forming a partition of V. X is said to be PO-independent (PO for Preserving Ordering) of Y in the context Z, denoted $I_{PO}(X, Z, Y)$, iff $\forall z \in D_Z, \forall y \in D_Y$:*

$$\forall x_i, x_j \in D_X, x_i \wedge z >_\Pi x_j \wedge z \textit{ iff } x_i \wedge y \wedge z >_\pi x_j \wedge y \wedge z. \tag{21}$$

Proposition 4 *If X is PO-independent of Y in the context Z, then X is also BP-independent of Y in the same context. The converse in not true.*

Counter-example 2 *Let us consider a ternary variable A and a binary variable B with the following plausibility distribution:*
$a_1 \wedge b_1 >_\pi a_2 \wedge b_1 >_\pi a_3 \wedge b_1 >_\pi a_1 \wedge b_2 >_\pi a_2 \wedge b_2 =_\pi a_3 \wedge b_2$.

We can check that A is BP-independent of B, but not PO-independent of B since the local plausibility relation relative to A is $a_1 >_\Pi a_2 >_\Pi a_3$. However, in the context b_2, we have $a_2 =_\Pi a_3$, thus the relation $I_{PO}(A, \emptyset, B)$ is false, since the ordering between a_2 and a_3 is not preserved in context b_2.

Note that this relation is not symmetrized as shown by the following counter example:

Counter-example 3 *Let us consider two binary variables A and B with the following ordering relation: $a_1 \wedge b_1 >_\pi a_1 \wedge b_2 >_\pi a_2 \wedge b_2 >_\pi a_2 \wedge b_1$.*

- *The local plausibility relation relative to A is $a_1 >_\Pi a_2$. Moreover, in the context b_1, we have $a_1 >_\Pi a_2$ and in the context b_2, we have $a_1 >_\Pi a_2$, thus, the relation $I_{PO}(A, \emptyset, B)$ is true since the ordering relative to the different instances of A is preserved for all instances of B.*

- *The local plausibility relation relative to B is $b_1 >_\Pi b_2$. However, in the context a_2, we have $b_2 >_\Pi b_1$, thus, the relation $I_{PO}(B, \emptyset, A)$ is false, since the ordering between b_1 and b_2 is not preserved in the context a_2.*

We denote I_{POS} the symmetrized version of I_{PO}; i.e. X is said to be POS-independent of Y in the context Z iff $\forall z \in D_Z, \forall y \in D_Y, \forall x \in D_X$:

$$(i)\ \forall x_i, x_j \in D_X, x_i \wedge z >_\Pi x_j \wedge z \text{ iff } x_i \wedge y \wedge z >_\pi x_j \wedge y \wedge z, \text{ and}$$

$$(ii)\ \forall y_k, y_l \in D_Y, y_k \wedge z >_\Pi y_l \wedge z \text{ iff } x \wedge y_k \wedge z >_\pi x \wedge y_l \wedge z. \tag{22}$$

The following proposition rewrites POS-independence in terms of **Acc**.

Proposition 5 *X is POS-independent of Y in the context Z iff:*

$$\forall D'_X \subseteq D_X, \forall D'_Y \subseteq D_Y \text{ such that } D'_X \neq \emptyset \text{ and } D'_Y \neq \emptyset \text{ and } \forall x, y, z$$

$$\mathbf{Acc}(x \wedge y \mid z \wedge D'_X \wedge D'_Y) = min(\mathbf{Acc}(x \mid z \wedge D'_X), \mathbf{Acc}(y \mid z \wedge D'_Y)). \tag{23}$$

From this rewriting, we deduce the following proposition:

Proposition 6 *If X is POS-independent of Y in the context Z, then X is also BPS-independent of Y in the same context. The converse in not true.*

5.2. ***Decompositional independence***

We now propose two classes of decompositional independencies, the first one based on belief decomposition and the second on remarkable plausibility relations.

5.2.1. *Belief decompositional independence*

The idea of this independence relation is to consider two variable sets X and Y as independent in context Z if for any instance z of Z, the acceptance of any instance $(x \wedge y)$ of (X, Y) is fully determined by the separate acceptance of x and of y. One way to relate the acceptance of $(x \wedge y)$ to the acceptance of x and the acceptance of y is:

Definition 4 *Let $\geq_\pi$ be a plausibility relation defined on $\Omega = D_V$ and consider three mutually disjoint subsets of variables X, Y and Z forming a partition of V. X and Y are said to be PT-independent (PT for Preserving Top elements) in the context Z, denoted by $I_{PT}(X, Z, Y)$, iff $\forall \phi_X \subseteq D_X, \forall \psi_Y \subseteq D_Y, \forall \xi_Z \subseteq D_Z$*

$$\mathbf{Acc}(\phi_X \wedge \psi_Y \mid \xi_Z) = min(\mathbf{Acc}(\phi_X \mid \xi_Z), \mathbf{Acc}(\psi_Y \mid \xi_Z)). \tag{24}$$

This definition is analogous to the one given in probability theory i.e. two variables A and B are independent if the probability over A and B is fully determined by $P(A)$ and $P(B)$ (i.e. $P(A \wedge B) = P(A) \cdot P(B)$).

Proposition 7 *X and Y are PT-independent in the context Z as soon as Definition 4 holds for all instances of X, Y and Z only, that is:*

$$\forall x, y, z, \mathbf{Acc}(x \wedge y \mid z) = min(\mathbf{Acc}(x \mid z), \mathbf{Acc}(y \mid z)). \tag{25}$$

It means that the set of plausible instances of a Cartesian product of domains is a Cartesian product. In particular, if any of the two sets $max(D_X)$ and $max(D_Y)$ contains a simple element then, obviously, X and Y are PT-independent. So PT-independent is a very weak definition of independence (see Figure 4 in Section 8).

The preservation of accepted beliefs implicitly implies the preservation of preferred instances but the converse is not true as stated by the following proposition.

Proposition 8 *If X is BPS-independent of Y in the context Z, then X and Y are also PT-independent. The converse in not true.*

Counter-example 4 *:* I_{PT} DOES NOT IMPLY I_{BPS}

Let us consider two binary variables A *and* B *with the following plausibility distribution:*$a_1 \wedge b_2 >_\pi a_2 \wedge b_2 >_\pi a_1 \wedge b_1 =_\pi a_2 \wedge b_1$.

We can check that A *and* B *are PT-independent, but not BPS independent.*

Using Propositions 8 and 6, we deduce that if X is POS-independent of Y in the context Z, then X and Y are also PT-independent and that the converse is not true.

5.2.2. *Decompositional independence of remarkable plausibility relations*

A natural way of defining decompositional independencies is to analyze the structure of the plausibility relation $\geq_\pi$. A plausibility relation is said to be decomposable w.r.t. X and Y in the context Z, iff $\geq_\pi$ is a function of the local orderings on $(X \cup Z)$ and $(Y \cup Z)$. The following introduces a well known principle, called *Pareto-principle*:

Definition 5 *Let* $\geq_\pi$ *be a comparative possibility relation and* u_i, v_i *be two instances (not necessarily different) of* A_i. *Let* $\overrightarrow{u} = (u_1, ..., u_n)$ *and* $\overrightarrow{v} = (v_1, ..., v_n)$ *be two vectors. Then,* $\overrightarrow{u}$ *is said to be weakly Pareto-preferred to* $\overrightarrow{v}$, *denoted by* $\overrightarrow{u} \geq_P \overrightarrow{v}$, *if and only if:* $\forall u_i, \forall v_i, i \in \{1, .., n\}, u_i \geq_\Pi v_i$. *Moreover,* $\overrightarrow{u}$ *is said to be strictly Pareto-preferred to* $\overrightarrow{v}$, *if and only if:* $\overrightarrow{u} \geq_P \overrightarrow{v}$ *and* $\exists i \in \{1, .., n\}$ *s.t.* $u_i >_\Pi v_i$.

In general $\geq_P$ is only a **partial** order. Since this paper deals with plausibility relations which are complete pre-orders, the following definition introduces a general class of plausibility relations which are compatible with the Pareto-principle:

Definition 6 *Let* X, Y *and* Z *be disjoint subsets of variables. A plausibility relation* $\geq_\pi$ *is said to be strictly Pareto-compatible (or* **monotonic***) along* X *and* Y *in the context* Z *if*
$\forall z \in D_Z, \forall x_i, x_j \in D_X, \forall y_k, y_l \in D_Y$, *we have:*
$(x_i \wedge z, y_k \wedge z) >_P (x_j \wedge z, y_l \wedge z)$ *implies* $(x_i \wedge y_k \wedge z) >_\pi (x_j \wedge y_l \wedge z)$

Well known example of orderings $\geq_\pi$ used in the comparative setting, which are Pareto-compatible are the *leximin* and the *leximax* orderings that we briefly present now[39].

Definition 7 *Let* $\overrightarrow{u} = \{u_1, ..., u_n\}$ *and* $\overrightarrow{v} = \{v_1, ..., v_n\}$ *be two vectors, and let* σ *and* τ *be two permutations of indices such that* $\forall i \in \{1, .., n\}, u_{\sigma(i)} >_\Pi u_{\sigma(i+1)}$ *and* $v_{\tau(i)} >_\Pi v_{\tau(i+1)}$. *Then,*
- $\overrightarrow{u}$ *is said to be leximin-preferred to* $\overrightarrow{v}$, *denoted by* $\overrightarrow{u} >_{leximin} \overrightarrow{v}$, *if and only if there exists* i *such that* $u_{\sigma(i)} >_\Pi v_{\tau(i)}$ *and* $\forall j > i, u_{\sigma(j)} =_\Pi v_{\tau(j)}$.
- $\overrightarrow{u}$ *is said to be leximin-equal to* $\overrightarrow{v}$, *denoted by* $\overrightarrow{u} =_{leximin} \overrightarrow{v}$, *if and only if* $\forall i$, $u_{\sigma(i)} =_\Pi v_{\tau(i)}$.

The leximin ordering is a natural extension of the minimum operator which has been used in different areas like in handling conflicts in knowledge bases[3,38], and in flexible constraint satisfaction problems[15,17,18].

Definition 8 *Let $\overrightarrow{u} = \{u_1, ..., u_n\}$ and $\overrightarrow{v} = \{v_1, ..., v_n\}$ be two vectors, and let σ and τ be two permutations of indices such that $\forall i \in \{1, .., n\}, u_{\sigma(i)} >_\Pi u_{\sigma(i+1)}$ and $v_{\tau(i)} >_\Pi v_{\tau(i+1)}$. Then,*
- $\overrightarrow{u}$ is said to be leximax-preferred to $\overrightarrow{v}$, denoted by $\overrightarrow{u} >_{leximax} \overrightarrow{v}$, if and only if there exists i such that $u_{\sigma(i)} >_\Pi v_{\tau(i)}$ and $\forall j < i, u_{\sigma(j)} =_\Pi v_{\tau(j)}$.
- $\overrightarrow{u}$ is said to be leximax-equal to $\overrightarrow{v}$, denoted by $\overrightarrow{u} =_{leximin} \overrightarrow{v}$, if and only if $\forall i$, $u_{\sigma(j)} =_\Pi v_{\tau(i)}$.

We now use these orderings to characterize plausibility relations:

1. A plausibility relation $\geq_\pi$ is said to be **Pareto-decomposable** along X and Y in the context Z, iff $\forall z \in D_Z, \forall x_i, x_j \in D_X, \forall y_k, y_l \in D_Y$, we have:
   $x_i \wedge y_k \wedge z \geq_\pi x_j \wedge y_l \wedge z$ **if and only if**
   $x_i \wedge z \geq_\Pi x_j \wedge z$ and $y_k \wedge z \geq_\Pi y_l \wedge z$.

   This definition is very strong, in the sense that $\geq_\pi$ is Pareto-decomposable along X and Y if one of the groups of variables is not informed as stated by the following proposition:

   Proposition 9 *A plausibility relation $\geq_\pi$ is Pareto-decomposable along X and Y iff one of the local plausibility relations on D_X or D_Y should be* uniform.

2. A plausibility relation $\geq_\pi$ is said to be **leximin-decomposable** along X and Y in the context Z, iff $\forall z \in D_Z, \forall x_i, x_j \in D_X, \forall y_k, y_l \in D_Y$, we have:

   - $x_i \wedge y_k \wedge z >_\pi x_j \wedge y_l \wedge z$ **if and only if**
     (i) $min(x_i \wedge z, y_k \wedge z) >_\Pi min(x_j \wedge z, y_l \wedge z)$ or
     (ii) $min(x_i \wedge z, y_k \wedge z) =_\Pi min(x_j \wedge z, y_l \wedge z)$ and
     $max(x_i \wedge z, y_k \wedge z) >_\Pi max(x_j \wedge z, y_l \wedge z)$.
   - $x_i \wedge y_k \wedge z =_\pi x_j \wedge y_l \wedge z$ **if and only if**
     $min(x_i \wedge z, y_k \wedge z) =_\Pi min(x_j \wedge z, y_l \wedge z)$ and
     $max(x_i \wedge z, y_k \wedge z) =_\Pi max(x_j \wedge z, y_l \wedge z)$.

3. A plausibility relation $\geq_\pi$ is said to be **leximax-decomposable** along X and Y in the context Z, iff $\forall z \in D_Z, \forall x_i, x_j \in D_X, \forall y_k, y_l \in D_Y$, we have:

   - $x_i \wedge y_k \wedge z >_\pi x_j \wedge y_l \wedge z$ **if and only if**
     (i) $max(x_i \wedge z, y_k \wedge z) >_\Pi max(x_j \wedge z, y_l \wedge z)$ or
     (ii) $max(x_i \wedge z, y_k \wedge z) =_\Pi max(x_j \wedge z, y_l \wedge z)$ and
     $min(x_i \wedge z, y_k \wedge z) >_\Pi min(x_j \wedge z, y_l \wedge z)$.
   - $x_i \wedge y_k \wedge z =_\pi x_j \wedge y_l \wedge z$ **if and only if**
     $min(x_i \wedge z, y_k \wedge z) =_\Pi min(x_j \wedge z, y_l \wedge z)$ and
     $max(x_i \wedge z, y_k \wedge z) =_\Pi max(x_j \wedge z, y_l \wedge z)$.

where $max(a,b) = \begin{cases} a \text{ if } a \geq_{\Pi} b \\ b \text{ otherwise} \end{cases}$

Definition 9 *X and Y are said to be* ***Pareto-independent*** *(resp.* ***leximin-independent, leximax-independent****) in the context Z, denoted I_{Pareto} (resp. $I_{leximin}$, $I_{leximax}$), if the plausibility relation $\geq_\pi$ is Pareto-decomposable (resp. leximin-decomposable, leximax-decomposable) along X and Y in the context Z.*

Proposition 10 *If X and Y are Pareto-independent in the context Z, then they are leximin-independent and leximax-independent. The converse is false and leximax independence is not comparable with leximin independence.*

Counter-example 5 : $I_{leximin}$ AND $I_{leximax}$ DO NOT IMPLY I_{Pareto} AND THEY ARE INCOMPRABLE

Let us consider the following plausibility relations pertaining to a binary variable A and ternary variable B:

$$a_1 \wedge b_1 >_\pi a_1 \wedge b_2 >_\pi a_2 \wedge b_1 >_\pi a_1 \wedge b_3 >_\pi a_2 \wedge b_2 >_\pi a_2 \wedge b_3,$$
$$a_1 \wedge b_1 >'_\pi a_1 \wedge b_2 >'_\pi a_2 \wedge b_1 >'_\pi a_2 \wedge b_2 >'_\pi a_1 \wedge b_3 >'_\pi a_2 \wedge b_3.$$

With $\geq_\pi$ we can check that A and B are leximax-independent but neither leximin-independent since $a_1 \wedge b_3 >_\pi a_2 \wedge b_2$ while $min(a_2, b_2) >_\Pi min(a_1, b_3)$, nor Pareto-independent since $a_1 \wedge b_3 >_\pi a_2 \wedge b_2$ while $b_2 >_\Pi b_3$.

In addition with $\geq'_\pi$ we can check that A and B are leximin-independent but neither leximax-independent since $a_2 \wedge b_2 >'_\pi a_1 \wedge b_3$ while $max(a_1, b_3) >'_\Pi max(a_2, b_2)$, nor Pareto-independent since $a_2 \wedge b_2 >'_\pi a_1 \wedge b_3$ while $a_1 >'_\Pi a_2$.

Proposition 11 *Pareto, leximin and leximax independence imply POS-independence. The converse is false.*

Counter-example 6 : I_{POS} DOES NOT IMPLY $I_{leximin}$, $I_{leximax}$ AND I_{Pareto}

Let A and B be two variables and $\geq_\pi$, $\geq'_\pi$ be the plausibility relations given in the previous counter example.
- with $\geq_\pi$, we can check that A is POS-independent of B but that these two variables are not leximin-independent since $a_1 \wedge b_3 >_\pi a_2 \wedge b_2$ while $min(a_2, b_2) >_\Pi min(a_1, b_3)$. Moreover, with $\geq'_\pi$ we can check that A is POS-independent of B but these two variables are not leximax-independent since $a_2 \wedge b_2 >'_\pi a_1 \wedge b_3$ while $max(a_1, b_3) >'_\Pi max(a_2, b_2)$.
- with the plausibility relation: $a_1 \wedge b_1 >_\pi a_1 \wedge b_2 >_\pi a_2 \wedge b_1 >_\pi a_2 \wedge b_2$, we can check that the relation $I_{POS}(A, \emptyset, B)$ is true contrary to $I_{Pareto}(A, \emptyset, B)$.

However, there are particular cases where the independence relations POS, leximin and leximax are equivalent:

- The first one concerns binary variables:

Proposition 12 *If A and B are binary variables then A is POS-independent of B in the context of a binary variable C if and only if they are leximin-independent and if and only if they are leximax-independent.*

- The second one concerns two-levels distributions:

Proposition 13 *If $\geq_\pi$ is a two-levels distribution, then X is POS-independent of Y in the context of Z if and only if they are leximin-independent and if and only if they are leximax-independent.*

6. Independence in scale-based possibility theory

In this section we recall well-known definitions of the independence relations which apply to a possibility distribution π[9,10,24,25,44,47]. Clearly this distribution induces a unique plausibility relation $\geq_\pi$ using (17); this will enable us to compare the independence relations introduced in this section with the ones in the previous sections. The comparison will be presented in Section 8.

6.1. *Possibilistic causal independence*

The idea in defining possibilistic causal independence relation based on the possibilistic conditioning is that X is considered as independent from Y in the context Z if for any instance $z \in D_Z$, the possibility degree of any $x \in D_X$ remains unchanged for any value $y \in D_Y$. More formally[9,10]:

$$\Pi(x \mid y \wedge z) = \Pi(x \mid z), \forall x, y, z. \tag{26}$$

Since possibility theory has two kinds of conditioning, this leads to two definitions of causal possibilistic independence:

- **Min-based independence relation** obtained by using the min-based conditioning (14) in (26). This form of independence called I_M is not symmetric i.e. $I_M(X, Z, Y) \neq I_M(Y, Z, X)$ where Z denotes the context variable, as pointed out by Fonck[25] and as shown by the following counter example.

  Counter-example 7 *Let us consider three binary variables A, B and C with the possibility distribution given in Table 2. We can check that $\pi(a \mid b \wedge c) = \pi(a \mid c), \forall a, b, c$ i.e. $I_M(A, C, B)$ is true but, $\pi(b_1 \mid a_1 \wedge c_1) = 1 \neq \pi(b_1 \mid c_1) = 0.7$ i.e. $I_M(B, C, A)$ is not true.*

  Table 2: Lack of symmetry property for I_M

a	b	c	$\pi(a \wedge b \wedge c)$
a_2	b_2	c_2	1
a_2	b_2	c_1	0.9
a_2	b_1	c_2	0.8
a_2	b_1	c_1	0.7
a_1	-	-	0.6

  Let us denote I_{MS} the symmetrized version[2] of I_M suggested by Fonck[24] (called MS-independence) $\forall x, y, z$:

  (i) $\Pi(x \mid_m y \wedge z) = \Pi(x \mid_m z)$ and

[2]In what follows the suffix S is used to denote the symmetrized version of non-symmetric relations.

$$\textbf{(ii)}\ \Pi(y \mid_m x \wedge z) = \Pi(y \mid_m z). \tag{27}$$

I_{MS} is a very strong relation since this MS-independence between two sets of variables X and Y requires full ignorance about one of them (uniform distribution) [9,10] i.e,
$\Pi(x) = 1, \forall x \in D_X$ or $\Pi(y) = 1, \forall y \in D_Y$.

- **Product independence relation** obtained by using the product-based conditioning (15) in (26)[9]. We can rewrite this form of independence using:

$$\Pi(x \wedge y \mid_p z) = \Pi(x \mid_p z) \cdot \Pi(y \mid_p z), \forall x, y, z, \tag{28}$$

or equivalently

$$\Pi(x \mid_p y \wedge z) = \Pi(x \mid_p z), \forall x, y, z. \tag{29}$$

Let us denote I_{Prod} the product based independence relation. The equivalence between (28) and (29) is true only for positive distributions. Moreover, product-based causal independence, contrary to min-based causal independence, is symmetric. This definition can be expressed in Spohn's ordinal function framework[43,42] using an appropriate transformation from integers to the unit scale $[0, 1]$. Indeed, this can be checked by showing that product-based conditioning is equivalent to Spohn's conditioning[21,19].

6.2. *Possibilistic decompositional independence: non-interactivity*

In the possibilistic framework, the standard decompositional independence between X and Y in the context Z is represented by the **non-interactivity** relation introduced by Zadeh[47], denoted by $I_{NI}(X, Z, Y)$ (NI for Non Interactivity) and defined by:

$$\Pi(x \wedge y \mid_m z) = min(\Pi(x \mid_m z), \Pi(y \mid_m z)), \forall x, y, z, \tag{30}$$

or equivalently by[24]:

$$\Pi(x \wedge y \wedge z) = min(\Pi(x \wedge z), \Pi(y \wedge z)), \forall x, y, z. \tag{31}$$

The following proposition relates existing independence relations in possibility theory.

Proposition 14 *MS-independence relation implies I_{NI}[26] and I_{Prod}.The converse is false. However, NI and Prod independence relations are incomparable.*

Counter-example 8 *: I_{Prod} AND I_{NI} DO NOT IMPLY I_{MS} AND THEY ARE INCOMPARABLE*

Let us consider two binary variables A and B with the possibility distributions given in Table 3. We can check that in π_1, the relation $I_{Prod}(A, \emptyset, B)$ is true contrary to $I_{MS}(A, \emptyset, B)$ and $I_{NI}(A, \emptyset, B)$. Moreover in π_2, the relation $I_{NI}(A, \emptyset, B)$ is true contrary to $I_{MS}(A, \emptyset, B)$ and $I_{Prod}(A, \emptyset, B)$.

Table 3: Relation between I_{Prod}, I_{NI} and I_{MS}

a	b	$\pi_1(a \wedge b)$	a	b	$\pi_2(a \wedge b)$
a_1	b_1	0.6	a_1	b_1	1
a_1	b_2	1	a_1	b_2	0.8
a_2	b_1	0.36	a_2	b_1	0.8
a_2	b_2	0.6	a_2	b_2	0.8

7. Commensurability and the decomposition of plausibility relations

In this section we study the *decomposition* of possibility orderings in the sense of some important independence relations. We will see that the commensurability property is crucial in the recomposition of joint distributions from marginal ones.

In the comparative setting, forming a joint possibility ordering from marginal ones is not immediate due to the absence of commensurability assumption between the different orderings. Indeed, different rankings are not expressing grades in the same scale and then it is impossible to compare the states, which makes it possible to build joint possibility relations.

In the possibilistic framework all the orderings are defined on the same scale e.g. [0, 1], which makes the composition of joint distributions from marginal ones easy. To illustrate the composition problem we will consider the case of the non-interactivity and leximin (resp. leximax) independence relations.

7.1. *Possibilistic non-interactivity*

The non-interactivity relation (see (30)) can be defined in a purely comparative setting as it is stated by the following proposition:

Proposition 15 *Let π be a possibility distribution. Let $\geq_\pi$ defined by $\omega \geq_\pi \omega'$ iff $\pi(\omega) \geq \pi(\omega')$. Then X and Y are NI-independent iff:*

$$x \wedge y \wedge z =_\Pi x \wedge z \text{ or } x \wedge y \wedge z =_\Pi y \wedge z, \forall x, y, z. \tag{32}$$

However, NI-independence is not interesting in a qualitative representation since it does not allow for the recomposition of a unique global plausibility relation from local orders defined on independent variables (due to the non-satisfaction of the commensurability property), as shown by the example below.

Example 6 *Let us consider two variables, relative to climatic conditions (CCdt) and physiological accidents (PAcc), such that:*
$D_{CCdt} = \{Bad(b), Good(g)\}$
$D_{PAcc} = \{Yes(y), No(n)\}$ *with the following local orderings:*

$$(i)\ b >_\Pi g \text{ and } (ii)\ y >_\Pi n.$$

There is no unique plausibility relation $\geq_\pi$ satisfying (i) and (ii) such that (CCdt) and (PAcc) are NI-independent. Indeed, it is sufficient to consider the two *plausibility relations* $\geq_\pi$ *and* $\geq'_\pi$*:*

$$b \wedge y >_\pi g \wedge y >_\pi b \wedge n =_\pi g \wedge n \text{ and } b \wedge y >'_\pi g \wedge y =' _\pi b \wedge n =' _\pi g \wedge n.$$

However, if the local orderings are encoded in possibility theory then there is a unique plausibility relation $\geq_\pi$ *using* $\pi(ccdt \wedge pacc) = min(\pi(ccdt), \pi(pacc)), \forall ccdt \in D_{CCdt}, \forall pacc \in D_{PAcc}$.

Indeed, if we encode the plausibility relation $\geq_\pi$ *by the possibility distribution given in Table 4, we obtain the local distributions on the two variables (CCdt) and (PAcc) given in Table 5. Then from* $\pi(ccdt)$ *and* $\pi(pacc)$ *we can recover (i) and (ii) in a unique manner using the min operator.*

Table 4: Decomposition by NI-independence

ccdt	pacc	$\pi(ccdt \wedge pacc)$
b	y	1
g	y	0.9
b	n	0.8
g	n	0.8

Table 5: Local distributions on CCdt and PAcc

ccdt	$\pi(ccdt)$	pacc	$\pi(pacc)$
b	1	y	1
g	0.9	n	0.8

The importance of the commensurability assumption also appears in fuzzy set based multicriteria aggregation, especially when defining connectives between fuzzy sets. For instance, French[27] questions the validity of the intersection definition of two fuzzy sets (using the minimum operator to define the membership function associated with the intersection) when no commensurability is assumed.

7.2. *Decomposition by leximin and leximax independence*

In the comparative setting, even if a plausibility relation is leximin or leximax decomposable, it can not be decomposed without loss of information, again due to the absence of commensurability assumption.

Example 7 *Let us consider two variables, relative to climatic conditions (CCdt) and maintenance (Maint), such that:*
$D_{CCdt} = \{Bad(bc), Good(gc)\}$
$D_{Maint} = \{Good(gm), Medium(mm), Weak(wm)\}$ *with the following plausibility relation* $\geq_\pi$ *which is leximin decomposable:*

$$bc \wedge gm >_\pi bc \wedge mm >_\pi gc \wedge gm >_\pi gc \wedge mm >_\pi bc \wedge wm >_\pi gc \wedge wm.$$

We can easily check that this plausibility relation can not be recovered from the induced local orders on (CCdt) and (Maint) given by:

$$(i)\ bc >_{\Pi} gc \text{ and } \ (ii)\ gm >_{\Pi} mm >_{\Pi} wm.$$

Indeed, it is sufficient to consider the following plausibility relation:

$$bc \wedge gm >'_{\pi} gc \wedge gm >'_{\pi} bc \wedge mm >'_{\pi} gc \wedge mm >'_{\pi} bc \wedge wm >'_{\pi} gc \wedge wm,$$

which satisfies (i) and (ii) and which is also leximin-decomposable.

Such a problem can be solved when considering the scale-based setting. Indeed, in this case the decomposition of leximin and leximax decomposable distributions is immediate since possibility degrees allow for the comparison between different states. In other terms, if the plausibility relation $\geq_\pi$ relative to any joint possibility distribution π is leximin or leximax decomposable then we can recover π from local distributions. Without a common scale, the use of the leximin or leximax does not allow the recovering of $\geq_\pi$.

Table 6: Decomposition by leximin-independence

ccdt	maint	$\pi(ccdt \wedge maint)$
bc	gm	1
bc	mm	0.9
gc	gm	0.8
gc	mm	0.7
bc	wm	0.3
gc	wm	0.2

Example 8 *Let π be a possibility distribution encoding the plausibility relation $\geq_\pi$ given in Example 7 (see Table 6). We can recover the plausibility relation behind π from the local distributions on (CCdt) and (Maint) (see Table 7) using the leximin ordering. Indeed, the use of the leximin on the local distributions provides the ordering relation relative to π i.e.*

$$bc \wedge gm >_{\pi} bc \wedge mm >_{\pi} gc \wedge gm >_{\pi} gc \wedge mm >_{\pi} bc \wedge wm >_{\pi} gc \wedge wm.$$

Moreover, if we have saved the numerical scale, namely (1,.9,.8,.7,.3,.2), we can recover the original distribution π. For instance, the state $bc \wedge gm$ corresponds to the possibility degree 1, $bc \wedge mm$ to 0.9 etc.

Table 7: Local distributions on CCdt and Maint

ccdt	$\pi(ccdt)$	maint	$\pi(maint)$
bc	1	gm	1
gc	0.8	mm	0.9
		wm	0.3

8. Comparative study

In Sections 5 and 6, we have established the different links existing, on the one hand, between scale-based possibilistic independence relations and, in the other hand, between independence relations expressed from plausibility orderings. This section compares all these relations. Namely, given a joint possibility distribution π, we will relate the relations I_{Pareto}, I_{POS}, $I_{leximin}$ and $I_{leximax}$ of Section 5 to the ones of Section 6 (i.e. I_{NI}, I_{MS} and I_{Prod}) by considering the plausibility relation $\geq_\pi$ induced from π by using (17).

Using Proposition 9, we can show the equivalence between MS and Pareto independence relations.

Proposition 16 *Let π be a possibility distribution, and $\geq_\pi$ be its associated plausibility relation. Then X and Y are MS-independent in π if and only if they are Pareto-independent in $\geq_\pi$.*

Proposition 17, shows that the M-independence implies the PO-independence.

Proposition 17 *If X is M-independent of Y in the context Z, then X is also PO-independent of Y in the same context. The converse is not true.*

Counter-example 9 *:* I_{PO} DOES NOT IMPLY I_M

Table 8: Relation between I_M and I_{PO}

a	b	$\pi(a \wedge b)$
a_1	b_1	1
a_1	b_2	0.8
a_2	b_2	0.7
a_2	b_1	0.5

Let us consider two binary variables A and B with the possibility distribution given in Table 8. We can check that the relation $I_{PO}(A, \emptyset, B)$ is true contrary to $I_M(A, \emptyset, B)$.

Propositions 18 and 19 relate Prod-independence to POS, leximin and leximax independencies.

Proposition 18 *If X and Y are Prod-independent in a strictly positive possibility distribution π, then X is POS-independent of Y in the plausibility relation induced by π. The converse is false.*

Counter-example 10 *:* I_{POS} DOES NOT IMPLY I_{Prod}

Let A and B be two variables with the strictly positive possibility distribution given in Table 9. We can check that the relation $I_{POS}(A, \emptyset, B)$ is true contrary to $I_{Prod}(A, \emptyset, B)$ since $\pi(a_2 \wedge b_3) = 0.5 \neq \Pi(a_2) \cdot \Pi(b_3) = 0.48$.

Counter-example 11 *Proposition 18 is false for non strictly positive possibility distributions. Indeed, let us consider two binary variables A and B with the non strictly positive possibility distribution given in Table 10. We can check that $I_{Prod}(A, \emptyset, B)$ is true, contrary to $I_{POS}(A, \emptyset, B)$.*

Table 9: Relation between I_{POS} and I_{Prod}

a	b	$\pi(a \wedge b)$
a_1	b_1	1
a_1	b_2	0.9
a_2	b_1	0.8
a_2	b_2	0.7
a_1	b_3	0.6
a_2	b_3	0.5

Table 10: Relation between I_{Prod} and I_{POS}

a	b	$\pi(a \wedge b)$
a_1	b_1	1
a_1	b_2	0
a_2	b_1	0
a_2	b_2	0

Proposition 19 *In the general case, the leximin and leximax independencies are incomparable with Prod-independence.*

Counter-example 12 *:* I_{Prod} INDEPENDENCE IMPLIES NEITHER $I_{leximin}$ NOR $I_{leximax}$ AND VICE VERSA

Table 11: Relation between I_{Prod}, $I_{leximin}$ and $I_{leximax}$

a	b	$\pi(a \wedge b)$
a_1	b_1	1
a_1	b_2	0.8
a_2	b_1	0.5
a_2	b_2	0.4
a_3	b_1	0.4
a_3	b_2	0.32

Let A and B be two variables,

- *with the possibility distribution given in Table 11, we can check that $I_{Prod}(A, \emptyset, B)$ is satisfied while $I_{leximin}(A, \emptyset, B)$ and $I_{leximax}(A, \emptyset, B)$ are false. Note that the product operator allows for compensation contrary to the leximin and leximax orderings.*

  For instance, if we have two pairs $(\Pi(x), \Pi(y))$ and $(\Pi(x'), \Pi(y'))$ such that $\Pi(x) > \Pi(x')$ and $\Pi(y') > \Pi(y)$, then $x \wedge y$ and $x' \wedge y'$ will be always strictly ranked using leximin and leximax principle. However, they can be equally ranked using the product operator since it may happen that $\Pi(x) \cdot \Pi(y) = \Pi(x') \cdot \Pi(y')$. This is the case in this example since $\Pi(a_2) = 0.5 > \Pi(a_3) = 0.4$ and $\Pi(b_1) = 1 > \Pi(b_2) = 0.8$ and $\Pi(a_2) \cdot \Pi(b_2) = \Pi(a_3) \cdot \Pi(b_1) = 0.4$.

Table 12: Relation between I_{Prod}, $I_{leximin}$ and $I_{leximax}$

a	b	$\pi(a \wedge b)$
a_1	b_1	1
a_1	b_2	0.9
a_2	b_1	0.8
a_1	b_3	0.5
a_2	b_2	0.3
a_2	b_3	0.2

- *with the possibility distributions given in Table 12, we can check that* $I_{leximax}(A, \emptyset, B)$ *is respected while* $I_{Prod}(A, \emptyset, B)$ *is false since* $\pi(a_2 \wedge b_2) = 0.3 \neq \Pi(a_2) \cdot \Pi(b_2) = 0.72$. *Moreover, with the possibility distribution given in Table 6, we can check that* $I_{leximin}(A, \emptyset, B)$ *is respected contrary to* $I_{Prod}(A, \emptyset, B)$ *since* $\pi(a_2 \wedge b_3) = 0.5 \neq \Pi(a_2) \cdot \Pi(b_3) = 0.48$.

Proposition 20 relates NI-independence to independence relations defined on plausibility relations.

Proposition 20 *Pareto independence implies NI-independence relation (since Pareto independence is equivalent to MS). Moreover, NI-independence implies PT-independence. However, this independence relation is incomparable with the other qualitative independence relations, namely the leximin, leximax, POS and BPS independencies.*

Counter-example 13 : I_{NI} IS INCOMPARABLE WITH $I_{leximin}$, $I_{leximax}$, I_{POS} AND I_{BPS} .

In the possibility distribution π_2 *given in Table 3, we can check that* $I_{NI}(A, \emptyset, B)$ *is true. However in the plausibility relation induced by* π_2 *(i.e.,* $a_1 \wedge b_1 >_\pi a_1 \wedge b_2 =_\pi a_2 \wedge b_1 =_\pi a_2 \wedge b_2$*) the relations* $I_{leximin}(A, \emptyset, B)$ *and* $I_{leximax}(A, \emptyset, B)$ *are false since* $a_1 \wedge b_2 =_\pi a_2 \wedge b_2$ *while* $max(a_1, b_2) >_\Pi max(a_2, b_2)$. *Moreover,* $I_{POS}(A, \emptyset, B)$ *is false since the local plausibility relation relative to A is* $a_1 >_\Pi a_2$ *while* $a_1 =_\Pi a_2$ *in the context of* b_1. *Lastly,* $I_{BPS}(A, \emptyset, B)$ *is false since* $\mathbf{Acc}(a_1 \mid b_2) = 0 \neq \mathbf{Acc}(a_1) = 1$.

In Table 12, we can check that the relation $I_{leximax}(A, \emptyset, B)$ *is respected contrary to* $I_{NI}(A, \emptyset, B)$ *since* $\pi(a_2 \wedge b_2) = 0.3 \neq min(\Pi(a_2), \Pi(b_2)) = min(0.8, 0.9) = 0.8$.

Lastly, in Table 9, we can check that $I_{leximin}(A, \emptyset, B)$ *is respected contrary to* $I_{NI}(A, \emptyset, B)$ *since* $\pi(a_2 \wedge b_2) = 0.7 \neq min(\Pi(a_2), \Pi(b_2)) = min(0.8, 0.9) = 0.8$. *Then we can deduce that in* π *the relations* $I_{POS}(A, \emptyset, B)$, $I_{BPS}(A, \emptyset, B)$ *and* $I_{PT}(A, \emptyset, B)$ *are true contrary to NI-independence since* $I_{leximin}$ *implies these three independence relations (from Propositions 8, 6 and 10).*

This counter example shows that if we start with a complete order, and map it to a scale (e.g. $[0, 1]$) then if X and Y are NI-independent (which implies that the distribution is decomposable with the minimum operator), then it is always possible to recompose the initial ordering from local ones defined on X and Y. This is possible because we can store the total pre-order by mapping it to a totally ordered scale. However, the case where the commensurability is crucial is when the expert provides local orders on X and Y and the fact that these sets of variables are NI-independent. Then, it is no longer possible to construct the global distribution.

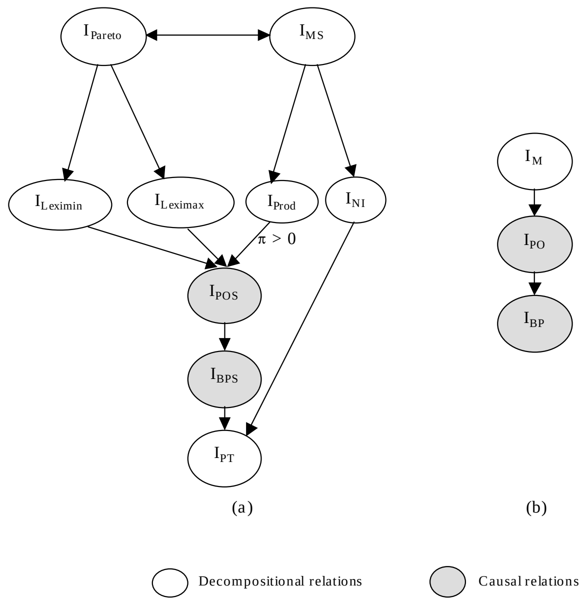

Figure 4: Links between symmetric (a) and non-symmetric (b) independence relations

Figure 4 (a) illustrates the links existing between the different symmetric independence relations. Figure 4 (b) concerns non-symmetric independence relations. The arrows show the inclusion between them. Note that I_{MS} and I_{Pareto} are the strongest independence relations since MS (or equivalently, Pareto) independence between two sets of variables imply a lack of information on one of them. However, I_{PT} is the weakest one, since it is sufficient to satisfy any independence relation in order to confirm that this relation is true. Finally, we remark that I_{NI} is implied by I_{MS} and I_{Pareto} and implies I_{PT} but it is incomparable with the other independence relations. Unsurprisingly, there are several decompositional independence relations according to the chosen decomposition mode. However, there is only one natural causality-oriented independence which is POS-independence relation. Fortunately, all of decompositional independence relations (except I_{NI}) are also meaningful from a causality point of view.

9. Conclusion

This paper relates notions of independence relations defined in the comparative setting (when only a complete pre-order is used) to the basic existing ones in the scale-based setting. Two kinds of independence have been investigated : causal and decompositional ones. We can observe that the independence relation which can be used for the decomposition of a joint distribution in possibility theory is not unique, contrary to probability theory where only the product-based independence can be used. In possibility theory, various forms of independence can be used. Another observation is that all of decompositional independence relations are also meaningful from a causality point of view, as clearly appears in Figure 4. Moreover, this paper has shown that the use of a common scale is crucial for decomposing distributions.

The notions of independence proposed in this paper extend previous works in default reasoning[4], and belief revision[13] on independence between events to the case of variables which are not necessarily binary. The notion of independence have been only compared to basic ones studied in literature. Comparison with other notion of numerical independence such as the ones introduced by de Campos and Huete [9,10] or Studený[44], is left for further research.

Another line for further research concerns logical counterpart to leximin and leximax independence relations in the possibilistic setting. Indeed, procedures for translating graph representations (defined from min-based and product-based conditional independence) into stratified possibilistic logic bases have been already proposed[5]. This is worth doing for leximin-based independence, which is stronger than NI-independence but still meaningful in a comparative setting.

Besides, the analysis of graphoid properties satisfied by the proposed independence relations is under study.

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Appendix

For the sake of simplicity the context appearing behind the conditioning bar is omitted in the following proofs.

Proof of Proposition 1 *Let us consider the possible values of* $\mathbf{Acc}_{\geq_\pi}(\phi \wedge \psi)$*:*

- $\mathbf{Acc}_{\geq_\pi}(\phi \wedge \psi) = 1 \Rightarrow \mathbf{Acc}_{\geq_\pi}(\phi) = 1$ *and* $\mathbf{Acc}_{\geq_\pi}(\psi) = 1$ *(using property 2)* *Hence,* $\mathbf{Acc}_{\geq_\pi}(\phi \wedge \psi) = min(\mathbf{Acc}_{\geq_\pi}(\phi), \mathbf{Acc}_{\geq_\pi}(\psi))$.
- $\mathbf{Acc}_{\geq_\pi}(\phi \wedge \psi) = 0 \Rightarrow \exists \phi', \psi'$ *s.t.* $\phi \wedge \psi =_\pi \phi' \wedge \psi'$ *(at least* $\phi' \neq_\Pi \phi$ *or* $\psi' \neq_\Pi \psi$*). If we assume that* $\phi' \neq_\Pi \phi$*, then* $\mathbf{Acc}_{\geq_\pi}(\phi) = 0$ *and* $\mathbf{Acc}_{\geq_\pi}(\psi) \geq 0$ $\Rightarrow \mathbf{Acc}_{\geq_\pi}(\phi \wedge \psi) = min(\mathbf{Acc}_{\geq_\pi}(\phi), \mathbf{Acc}_{\geq_\pi}(\psi))$.
- $\mathbf{Acc}_{\geq_\pi}(\phi \wedge \psi) = -1$. *Using* $\mathbf{Acc}_{\geq_\pi}(\phi \wedge \psi) \neq min(\mathbf{Acc}_{\geq_\pi}(\phi), \mathbf{Acc}_{\geq_\pi}(\psi))$*, we deduce that (i1)* $\mathbf{Acc}_{\geq_\pi}(\phi) \geq 0$ *and (i2)* $\mathbf{Acc}_{\geq_\pi}(\psi) \geq 0$.
  If we assume that $\mathbf{Acc}_{\geq_\pi}(\phi) = 1$*, then* $\phi >_\pi \neg\phi$*. Thus, we can distinguish two cases:*
  - $\phi \wedge \neg\psi$ *is more plausible than* $\neg\phi \wedge \psi$ *and* $\neg\phi \wedge \neg\psi$ $\Rightarrow \mathbf{Acc}_{\geq_\pi}(\psi) = -1$ *which contradicts (i2).*

– $\phi \wedge \psi$ *is more plausible than* $\neg\phi \wedge \psi$ *and* $\neg\phi \wedge \neg\psi$. *In this case* $\phi \wedge \neg\psi$ *is more plausible than* $\phi \wedge \psi$ *(since* $\mathbf{Acc}_{\geq_\pi}(\phi \wedge \psi) = -1$*)*
$\Rightarrow \mathbf{Acc}_{\geq_\pi}(\psi) = -1$ *which contradicts (i2).*

Proof of Proposition 2 *The possible values that* $\mathbf{Acc}(\phi \wedge \psi)$ *can take on are:*

- $\mathbf{Acc}(\phi \wedge \psi) = 1 \Rightarrow \phi \wedge \psi >_\Pi \neg(\phi \wedge \psi) = max(\neg\phi \wedge \psi, \phi \wedge \neg\psi, \neg\phi \wedge \neg\psi)$ *and hence* $\phi \wedge \psi >_\Pi \neg\phi \wedge \psi$ *which implies* $\mathbf{Acc}(\phi \mid \psi) = 1$.
- $\mathbf{Acc}(\phi \wedge \psi) = -1 \Rightarrow \phi \wedge \psi <_\Pi \neg(\phi \wedge \psi)$.
  $\Rightarrow max(\Omega) \subseteq (\neg\phi \wedge \psi) \vee (\phi \wedge \neg\psi) \vee (\neg\phi \wedge \neg\psi)$,
  - *if* $\neg\phi \wedge \psi \subseteq max(\Omega)$ *then* $\mathbf{Acc}(\phi \mid \psi) = -1$,
  - *else* $max(\Omega) \subseteq \neg\psi$, $\psi <_\Pi \neg\psi$ *which implies that* $\mathbf{Acc}(\psi) = -1$.
- $\mathbf{Acc}(\phi \wedge \psi) = 0 \Rightarrow \phi \wedge \psi =_\Pi \neg(\phi \wedge \psi)$
  $\Rightarrow \phi \wedge \psi \subseteq max(\Omega)$ *and* $(\neg\phi \wedge \psi) \vee (\phi \wedge \neg\psi) \vee (\neg\phi \wedge \neg\psi) \cap max(\Omega) \neq \emptyset$,
  - *if* $\neg\phi \wedge \psi \subseteq max(\Omega) \Rightarrow \mathbf{Acc}(\phi \mid \psi) = 0$ *(since* $\phi \wedge \psi =_\Pi \neg\phi \wedge \psi$*)*
  and $\mathbf{Acc}(\psi) \geq 0$ *(since* $\phi \wedge \psi \subseteq \psi$*)*,
  - *else* $max(\Omega) \subseteq \neg\phi \Rightarrow \mathbf{Acc}(\phi \mid \psi) = 1$ *(since* $\phi \wedge \psi >_\Pi \neg\phi \wedge \psi$*)*
  and $\mathbf{Acc}(\psi) = 0$ *(since* $\psi =_\Pi \max_{\omega \in [\psi]} \omega =_\Pi \neg\psi =_\Pi \max_{\omega \in [\neg\psi]} \omega$*)*.

Proof of Proposition 3 *The independence relation* $I_{BP}(X, Z, Y)$ *implies this Proposition trivially by replacing* ϕ_X *by* x, ψ_Y *by* y *and* ξ_Z *by* z *in (18). Thus it is enough to prove that if* $\forall x, y, z, \mathbf{Acc}(x \mid y \wedge z) = \mathbf{Acc}(x \mid z)$ *then* $I_{BP}(X, Z, Y)$. *Since* ***Acc*** *on instances of* X *characterizes the acceptance function on subsets of* D_X, *it follows that* $\forall y, z$ $\boldsymbol{Acc}(\phi_X \mid y \wedge z) = \boldsymbol{Acc}(\phi_X \mid z)$. *Now it is obvious that plausibility relations* $\psi_1 >_\Pi \xi_1$ *and* $\psi_2 >_\Pi \xi_2$ *imply* $\psi_1 \vee \psi_2 >_\Pi \xi_1 \vee \xi_2$ *and the same for* $=_\Pi$. *Hence, from* $\phi_X \wedge y \wedge z >_\Pi \neg\phi_X \wedge y \wedge z$ $\forall y \in \psi_Y, \forall z \in \xi_Z$ *imply* $\phi_X \wedge \psi_Y \wedge \xi_Z >_\Pi \neg\phi_X \wedge \psi_Y \wedge \xi_Z$, *and the same with* $=_\Pi$. *Hence,* $I_{BP}(X, Z, Y)$ *holds.*

Proof of Proposition 4 *We can distinguish three cases:*

- $\mathbf{Acc}(x) = 1$, *this means that* $\forall x', x >_\Pi x'$. *Therefore,* $\forall x', x \wedge y >_\pi x' \wedge y$, $\mathbf{Acc}(x \mid y) = 1$.
- $\mathbf{Acc}(x) = -1$, *this means that* $\exists x'$ *s.t.* $x' >_\Pi x$. *Then using PO-independence we deduce that* $\exists x'$ *s.t.* $x' \wedge y >_\pi x \wedge y$ *which implies* $\mathbf{Acc}(x \mid y) = -1$
- $\mathbf{Acc}(x) = 0$, *this means that* $\exists x'$ *s.t.* $x =_\Pi x'$ *and* $\nexists x''$ *s.t.* $x'' >_\Pi x$
  $\Rightarrow x \wedge y =_\pi x' \wedge y$ *and* $\nexists x''$ *s.t.* $x'' \wedge y =_\pi x \wedge y$
  $\Rightarrow \mathbf{Acc}(x \mid y) = 0$

Proof of Proposition 5 *Let* $\phi = D'_X$ *and* $\psi = D'_Y$.
- *Assume that (i)* $\mathbf{Acc}(x \wedge y \mid \phi \wedge \psi) = min(\mathbf{Acc}(x \mid \phi), \mathbf{Acc}(y \mid \psi))$
then $\forall x, x', \forall y$, *we have:*
$\mathbf{Acc}(x \wedge y \mid \{x, x'\} \wedge y) = min(\mathbf{Acc}(x \mid \{x, x'\}), \mathbf{Acc}(y \mid y))$

$\Leftrightarrow \mathbf{Acc}(x \mid \{x, x'\} \wedge y) = \mathbf{Acc}(x \mid \{x, x'\})$
since $\mathbf{Acc}(y \mid y) = 1$, *and* $\mathbf{Acc}(x \wedge y \mid \{x, x'\} \wedge y) = \mathbf{Acc}(x \mid \{x, x'\} \wedge y)$.
If $\mathbf{Acc}(x \mid \{x, x'\}) = 1$ *(resp. 0,-1), then* $x >_\Pi x'$ *(resp.* $x =_\Pi x', x' >_\Pi x$*)*
$\Rightarrow \mathbf{Acc}(x \mid \{x, x'\} \wedge y) = 1$ *(resp. 0,-1)*
$\Rightarrow x \wedge y >_\pi x' \wedge y$ *(resp.* $x \wedge y =_\pi x' \wedge y, x' \wedge y >_\pi x \wedge y$*).*
Therefore, X *is PO-independent of* Y*, Moreover* Y *is PO-independent of* X *since (i) is obviously symmetric. Hence,* X *is POS-independent of* Y*.*

- Let us show the converse.
Assume X *and* Y *are POS-independent.*
First suppose that $\mathbf{Acc}(x \mid \phi) = -1$*. Then* $\exists x' \in \phi$ *such that* $x' >_\Pi x$
$\Rightarrow x' \wedge y >_\pi x \wedge y$ *(since* I_{POS} *is true)*
$\Rightarrow \mathbf{Acc}(x \wedge y \mid \phi \wedge \psi) = -1$ *(by definition).*
The same conclusion holds if $\mathbf{Acc}(y \mid \psi) = -1$*.*
Now, we have three remaining cases (other cases, are obtained by symmetry):

- $\mathbf{Acc}(x \mid \phi) = 1$ *and* $\mathbf{Acc}(y \mid \psi) = 1$*.*
  Assume that $\mathbf{Acc}(x \wedge y \mid \phi \wedge \psi) \neq 1$ *(i.e. 0 or -1)*
  $\Rightarrow \exists x' \in \phi, y' \in \psi$ *such that* $x' \wedge y' \geq_\pi x \wedge y$
  Moreover, $x >_\Pi x'$ *(since* $\mathbf{Acc}(x \mid \phi) = 1$*)*
  $\Rightarrow x \wedge y >_\pi x' \wedge y$ *(since* I_{POS} *is true)*
  $\Rightarrow x' \wedge y' \geq_\pi x \wedge y >_\pi x' \wedge y$
  $\Rightarrow x' \wedge y' >_\pi x' \wedge y$
  $\Rightarrow y' \geq_\Pi y$ *(since* I_{POS} *is true). Hence contradiction with* $\mathbf{Acc}(y \mid \psi) = 1$*.*

- $\mathbf{Acc}(x \mid \phi) = 0$ *and* $\mathbf{Acc}(y \mid \psi) = 1$*.*
  Assume that $\mathbf{Acc}(x \wedge y \mid \phi, \psi) \neq 0$ *(i.e. -1 or 1), then we can consider two subcases:*
  - $\mathbf{Acc}(x \wedge y \mid \phi \wedge \psi) = -1 \Rightarrow \exists x' \in \phi, y' \in \psi$ *such that* $x' \wedge y' >_\pi x \wedge y$
  Moreover, $y >_\Pi y'$ *(since* $\mathbf{Acc}(y \mid \psi) = 1$*)*
  $\Rightarrow x \wedge y >_\pi x \wedge y'$ *(since* I_{POS} *is true)*
  $\Rightarrow x' \wedge y' >_\pi x \wedge y'$
  $\Rightarrow x' >_\Pi x$ *(since* I_{POS} *is true). Hence contradiction with* $\mathbf{Acc}(x \mid \phi) = 0$*.*
  - $\mathbf{Acc}(x \wedge y \mid \phi \wedge \psi) = 1 \Rightarrow \forall x' \in \phi, \forall y' \in \psi, x \wedge y >_\pi x' \wedge y'$
  $\Rightarrow \forall x', x \wedge y >_\pi x' \wedge y \Rightarrow x >_\Pi x'$ *(since* I_{POS} *is true).*
  Hence contradiction with $\mathbf{Acc}(x \mid \phi) = 0$*.*

- $\mathbf{Acc}(x \mid \phi) = 0$ *and* $\mathbf{Acc}(y \mid \psi) = 0$*.*
  Assume that $\mathbf{Acc}(x \wedge y \mid \phi \wedge \psi) \neq 0$ *(i.e. -1 or 1), then we can consider two subcases:*
  - $\mathbf{Acc}(x \wedge y \mid \phi \wedge \psi) = -1 \Rightarrow \exists x' \in \phi, y' \in \psi$ *such that* $x' \wedge y' >_\pi x \wedge y$
  Moreover, $y =_\Pi y'$ *(since* $\mathbf{Acc}(y \mid \psi) = 0$*)*
  $\Rightarrow x \wedge y =_\pi x \wedge y'$ *(since* I_{POS} *is true)*
  $\Rightarrow x' \wedge y' >_\pi x \wedge y'$
  $\Rightarrow x' >_\Pi x$ *(since* I_{POS} *is true). Hence contradiction with* $\mathbf{Acc}(x \mid \phi) = 0$*.*

- $\mathbf{Acc}(x \wedge y \mid \phi \wedge \psi) = 1 \Rightarrow \forall x' \in \phi, \forall y' \in \psi, x \wedge y >_\pi x' \wedge y'$
$\Rightarrow \forall x', x \wedge y >_\pi x' \wedge y \Rightarrow \forall x', x >_\Pi x'$ *(since I_{POS} is true).*
Hence contradiction with $\mathbf{Acc}(x \mid \phi) = 0$.

Proof of Proposition 6 *Let $D'_X = D_X$ and $D'_Y = \{y\}$ in (23), then we obtain:*
$\mathbf{Acc}(x \wedge y \mid D_X, \{y\}) = min(\mathbf{Acc}(x \mid D_X), \mathbf{Acc}(y \mid \{y\}))$
$\Leftrightarrow \mathbf{Acc}(x \mid \{y\}) = \mathbf{Acc}(x)$ *(since $\mathbf{Acc}(x \wedge y \mid D_X, \{y\}) = \mathbf{Acc}(x \mid \{y\})$,*
$\mathbf{Acc}(x \mid D_X) = \mathbf{Acc}(x)$ and $\mathbf{Acc}(y \mid \{y\}) = 1$)
which leads to case (i) of (20). The case (ii) of (20) is obtained by symmetry by letting $D'_X = \{x\}$ and $D'_Y = D_Y$ in (23).

Proof of Proposition 7 *To show this proof, note that Equation (25) does not hold only if $\mathbf{Acc}(x \wedge y \mid z) = -1$ and $\mathbf{Acc}(x \mid z) = \mathbf{Acc}(y \mid z) = 0$ due to the properties of **Acc** given in Subsection 3.2.*
This is equivalent to assuming that in any context z, $card(max(D_X)) > 1$, $card(max(D_Y)) > 1$, and $x \wedge y \notin max(D_X \times D_Y)$ for some $x \in max(D_X)$ and some $y \in max(D_Y)$.
So (25) means that $max(D_X \times D_Y) = max(D_X) \times max(D_Y)$ in any context z. Hence, $\forall \xi_Z \subseteq D_Z, max(D_X \times D_Y) \times D_Z = max(D_X) \times max(D_Y) \times D_Z$. However, PT independence does not apply only if:
$\mathbf{Acc}(\phi_X \wedge \psi_Y \mid \xi_Z) = -1, \mathbf{Acc}(\phi_X \mid \xi_Z) = 0,$ *and* $\mathbf{Acc}(\psi_Y \mid \xi_Z) = 0$
which is equivalent to say that:
$\phi_X \wedge \neg\psi_Y \wedge \xi_Z =_\pi \neg\phi_X \wedge \psi_Y \wedge \xi_Z \geq_\pi \neg\phi_X \wedge \neg\psi_Y \wedge \xi_Z >_\pi \phi_X \wedge \psi_Y \wedge \xi_Z$.
However, it implies that $max(D_X)$ overlaps ϕ_X and $\neg\phi_X$, $max(D_Y)$ overlaps ψ_Y and $\neg\psi_Y$, and $\forall x \wedge y \in \phi_X \wedge \psi_Y \wedge max(D_X) \times max(D_y)$, $x \wedge y \notin max(D_X \times D_Y)$. Hence, we have proved that in context ξ_Z, the equality $\mathbf{Acc}(x \wedge y \mid \xi_Z) = min(\mathbf{Acc}(x \mid \xi_Z), \mathbf{Acc}(y \mid \xi_Z))$ does not hold. It implies that $\exists z \in \xi_Z$, such that (25) does not hold. So (25) implies PT-independence.

Proof of Proposition 8 *Suppose that this Proposition is false i.e. X and Y are not PT-independent. Hence $\exists x, \exists y$, such that $\mathbf{Acc}(x \wedge y) \neq min(\mathbf{Acc}(x), \mathbf{Acc}(y))$*
$\Rightarrow min(\mathbf{Acc}(x \mid y), \mathbf{Acc}(y)) \neq min(\mathbf{Acc}(x), \mathbf{Acc}(y))$ *(From Proposition 2).*
Hence $\mathbf{Acc}(x \mid y) \neq \mathbf{Acc}(x)$.
So X and Y are not PO-independent. Hence they cannot be BPS-independent.

Proof of Proposition 9 *Suppose that none of the distributions on X and Y is uniform, that is $\exists x, x', \exists y, y'$, s.t. $x >_\Pi x'$ and $y >_\Pi y'$. Then the two states $x \wedge y'$ and $x' \wedge y$ are not comparable. Indeed:*
- *if $x \wedge y' \geq_\pi x' \wedge y$, this relation contradicts the Pareto-ordering since $y' \not\geq_\Pi y$,*
- *if $x' \wedge y \geq_\pi x \wedge y'$, this relation contradicts the Pareto-ordering since $x' \not\geq_\Pi x$.*
This result contradicts the assumption that $\geq_\pi$ encodes a complete preorder.
Let us show the converse. Assume that the distribution on D_X is uniform but $\geq_\pi$ is not Pareto-independent. This means that $\exists x, x', \exists y, y'$ such that $x \wedge y \geq_\pi x' \wedge y'$ but

$y <_{\Pi} y'$. *This implies that* $x' \wedge y' =_{\pi} x \wedge y' >_{\pi} x \wedge y$ *(since* $x = \Pi x'$ *and* $y <_{\Pi} y'$*). Hence a contradiction.*

Proof of Proposition 10 - PROOF THAT I_{Pareto} IMPLIES $I_{leximin}$. *Suppose that* X *and* Y *are Pareto-independent but not leximin-independent, then we can distinguish two cases:*
- ***Case 1***: $\exists x, y, \exists x', y'$, *s.t.* $x \wedge y >_{\pi} x' \wedge y'$ *but*
$min(x, y) <_{\Pi} min(x', y')$, *or*
$min(x, y) =_{\Pi} min(x', y')$ *and* $max(x, y) \leq_{\Pi} max(x', y')$
Since Pareto-independence is respected $x \wedge y >_{\pi} x' \wedge y'$
$\Rightarrow x >_{\Pi} x'$ *and* $y >_{\Pi} y'$
$\Rightarrow min(x, x') >_{\Pi} min(y, y')$ *and* $max(x, x') >_{\Pi} max(y, y')$.
Hence contradiction.
- ***Case 2***: $\exists x, y, \exists x', y'$, *s.t.* $x \wedge y =_{\pi} x' \wedge y'$ *but*
$min(x, y) \neq_{\Pi} min(x', y')$, *or* $max(x, y) \neq_{\Pi} max(x', y')$
Since Pareto-independence is respected $x \wedge y =_{\pi} x' \wedge y'$
$\Rightarrow x =_{\Pi} x'$ *and* $y =_{\Pi} y'$
$\Rightarrow min(x, y) =_{\Pi} min(x', y')$, *and* $max(x, y) =_{\Pi} max(x', y')$
A contradiction again.
- PROOF THAT I_{Pareto} IMPLIES $I_{leximax}$. *This can be done in the same manner as* $I_{Pareto}(X, Z, Y) \Rightarrow I_{leximax}(X, Z, Y)$.

Proof of Proposition 11 - PROOF THAT $I_{leximin}$ IMPLIES I_{POS}. *Suppose that not i.e.* X *and* Y *are leximin but not POS-independent then we can distinguish two possible situations (other cases, are obtained by symmetry):*
- ***Case 1***: $\exists x, y, \exists x', y'$ *s.t.* $x \geq_{\Pi} x'$ *and* $x \wedge y <_{\pi} x' \wedge y$
Since leximin-independence is respected, $x \wedge y <_{\pi} x' \wedge y$
$$\Rightarrow \begin{cases} min(x, y) <_{\Pi} min(x', y) \text{ or} \\ min(x, y) =_{\Pi} min(x', y) \text{ and } max(x, y) <_{\Pi} max(x', y) \end{cases}$$
$\Rightarrow x <_{\Pi} x'$. *Hence contradiction.*
- ***Case 2***: $\exists x, y, \exists x', y'$ *s.t.* $x >_{\Pi} x'$ *and* $x \wedge y \leq_{\pi} x' \wedge y$
Since leximin-independence is respected $x \wedge y \leq_{\pi} x' \wedge y$
$\Rightarrow min(x, y) \leq_{\Pi} min(x', y)$ *and* $max(x, y) \leq_{\Pi} max(x', y)$
$\Rightarrow x \leq_{\Pi} x'$. *Hence contradiction.*

Proof of Proposition 12 - PROOF THAT I_{POS} IS EQUIVALENT TO $I_{leximin}$ IN THE BINARY CASE. *It is enough to prove that POS-independence implies leximin-independence. Suppose that* A *and* B *are POS but not leximin-independent i.e.* $\exists a, a', \exists b, b'$ *s.t. (i)* $a \wedge b >_{\pi} a' \wedge b'$ *but the relation* $a \wedge b >_{leximin} a' \wedge b'$ *is false. This may happen in two cases:*
- ***Case 1***: $min(a, b) <_{\Pi} min(a', b') \Rightarrow min(a, b) <_{\Pi} a'$ *and* $min(a, b) <_{\Pi} b'$
Suppose that $a \leq_{\Pi} b$ *then we have* $a <_{\Pi} a'$ *and* $a <_{\Pi} b'$
From the POS-independence, $a <_{\Pi} a'$ *implies:*

(ii) $a \wedge b <_\pi a' \wedge b$ *and (iii)* $a \wedge b' <_\pi a' \wedge b'$
From (i),(ii) and (iii) we have $a \wedge b' <_\pi a' \wedge b' <_\pi a \wedge b <_\pi a' \wedge b$
which contradicts $a <_\pi b'$.
- ***Case 2**:* $min(a,b) =_\Pi min(a',b')$ *and* $max(a,b) \leq_\Pi max(a',b')$
Suppose that $a \leq_\Pi b$ *then we have* $a =_\Pi min(a',b')$ *and* $b \leq_\Pi max(a',b')$
Suppose now that $a' \leq_\Pi b'$ *then we have* $a =_\Pi a'$ *and* $b \leq_\Pi b'$.
From the POS-independence, $b \leq_\Pi b'$ *implies:*
(ii) $a \wedge b \leq_\pi a \wedge b'$ *and (iii)* $a' \wedge b \leq_\pi a' \wedge b'$
From (i),(ii) and (iii) we have $a' \wedge b \leq_\pi a' \wedge b' <_\pi a \wedge b \leq_\pi a \wedge b'$
which contradicts $a =_\Pi a'$.
- PROOF THAT I_{POS} IS EQUIVALENT TO $I_{leximax}$ IN THE BINARY CASE. *This can be done in the same manner as* I_{POS} *is equivalent to* $I_{leximin}$.

Proof of Proposition 13 - PROOF THAT I_{POS} IS EQUIVALENT TO $I_{leximin}$ IF WE HAVE TWO-LEVELS DISTRIBUTIONS.
It is enough to prove that POS-independence implies leximin-independence. Suppose that A *and* B *are POS but not leximin-independent i.e.* $\exists a, a', \exists b, b'$ *s.t. (i)* $a \wedge b >_\pi a' \wedge b'$ *but the relation* $a \wedge b >_{leximin} a' \wedge b'$ *is false. This may happen in two cases:*
- ***Case 1**:* $min(a,b) <_\Pi min(a',b') \Rightarrow min(a,b) <_\Pi a'$ *and* $min(a,b) <_\Pi b'$
Suppose that $a \leq_\Pi b$ *then from Case 1 of Proof 12, we have*
$a \wedge b' <_\pi a' \wedge b' <_\pi a \wedge b <_\pi a' \wedge b$ *which contradicts the fact that the distributions have only two levels.*
- ***Case 2**:* $min(a,b) =_\Pi min(a',b')$ *and* $max(a,b) \leq_\Pi max(a',b')$
Suppose that $a \leq_\Pi b$ *then we have* $a =_\Pi min(a',b')$ *and* $b \leq_\Pi max(a',b')$
Suppose now that $a' \leq_\Pi b'$ *then we have* $a =_\Pi a'$ *and* $b \leq_\Pi b'$.
From the POS-independence, $a =_\Pi a'$ *and* $b \leq_\Pi b'$ *imply respectively:*
$a \wedge b =_\pi a' \wedge b$ *and* $a \wedge b' =_\pi a' \wedge b'$
$a \wedge b \leq_\pi a \wedge b'$ *and* $a' \wedge b \leq_\pi a' \wedge b'$
Moreover, from (i) we deduce that $a \wedge b$ *is among the top elements (i.e.* $\boldsymbol{Acc}(a \wedge b) = 1$*) since we have two-levels distributions. Thus* $a \wedge b =_\pi a \wedge b' =_\pi a' \wedge b'$.
Hence contradiction.
- PROOF THAT I_{POS} IS EQUIVALENT TO $I_{leximax}$ IF WE HAVE TWO-LEVELS DISTRIBUTIONS. *This can be done in the same manner as* I_{POS} *is equivalent to* $I_{leximin}$.

Proof of Proposition 14 *Suppose that* $\exists x, \exists y$, *such that (i)* $\Pi(x \wedge y) \neq \Pi(x) \cdot \Pi(y)$. *Suppose that the distribution on* X *is uniform (from Proposition 16), then* $\Pi(x) = 1$, *thus* $\Pi(x \wedge y) < 1$ *(since* $\Pi(x \wedge y) < \Pi(y)$*). Hence,* $\Pi(y \mid x) = \Pi(x \wedge y)$.
Moreover $\Pi(y \mid x) = \Pi(y)$ *since* X *and* Y *are MS-independent.*
Hence contradiction.

Proof of Proposition 15 *The proof is immediate since* X *and* Y *are NI-independent means* $\Pi(x \wedge y \wedge z) = min(\Pi(x \wedge y), \Pi(x \wedge z))$. *Namely,* $\Pi(x \wedge y \wedge z) = \Pi(x \wedge y)$ *or*

$\Pi(x \wedge y \wedge z) = \Pi(y \wedge z)$ *which is equivalent to:* $x \wedge y \wedge z =_\Pi x \wedge y$ *or* $x \wedge y \wedge z =_\Pi y \wedge z$ *(since if* $\omega \geq_\pi \omega'$ *iff* $\pi(\omega) \geq \pi(\omega')$ *then* $\phi \geq_\Pi \psi'$ *iff* $\Pi(\phi) \geq \Pi(\psi')$*)*

Proof of Proposition 16 *It is obvious that if X and Y are MS-independent then they are Pareto-independent from Propositions 9 and the fact that MS-independence implies that one of the local plausibility relations on D_X or D_Y should be uniform. We now prove that if X and Y are Pareto-independent then they are MS-independent in π. Suppose that $\exists x, \exists y$, such that $\Pi(x \mid y) \neq \Pi(x)$, then we can distinguish two cases:*

- *Case 1*: $\Pi(x) = 1 \Rightarrow \Pi(x \mid y) < 1$

$$\Rightarrow \begin{cases} \Pi(x \wedge y) < \Pi(y) \text{ (conditioning definition) and} \\ \Pi(x \wedge y) < \Pi(x) \text{ (indeed } \Pi(x \mid y) = \Pi(x \wedge y) < 1 = \Pi(x)) \end{cases}$$

$\Rightarrow \exists x', \exists y'$, *s.t.* $\Pi(x \wedge y) < \Pi(x' \wedge y)$ *and* $\Pi(x \wedge y) < \Pi(x \wedge y')$

$\Rightarrow \exists x', \exists y'$, *s.t.* $x \wedge y <_\pi x' \wedge y$ *and* $x \wedge y <_\pi x \wedge y'$

Since the plausibility relation $\geq_\pi$ is Pareto-decomposable, $\exists x', \exists y'$, s.t. $x <_\Pi x'$ and $y <_\Pi y'$ which contradicts proposition 9.

- *Case 2*: $\Pi(x) \neq 1$, *then the two possible situations are:*

- $\Pi(x \mid y) = 1$
  $\Rightarrow \Pi(x \wedge y) = \Pi(y)$ *(conditioning definition)*
  $\Rightarrow \forall x', \Pi(x \wedge y) \geq \Pi(x' \wedge y)$.
  $\Rightarrow \forall x', x \wedge y \geq_\pi x' \wedge y$.
  Since the plausibility relation $\geq_\pi$ is Pareto-decomposable, we have $\forall x', x \geq_\Pi x'$ thus x is the top element. However, from $\Pi(x) < 1$ we deduce that x is not the top element. Hence contradiction.

- $\Pi(x \mid y) \neq \Pi(x) < 1$
  $$\Rightarrow \begin{cases} \Pi(x \wedge y) < \Pi(y) \text{ (conditioning definition) and} \\ \Pi(x \wedge y) < \Pi(x) \text{ (indeed we have by definition } \Pi(x \wedge y) \leq \Pi(x) \\ \quad \text{moreover } \Pi(x \mid y) = \Pi(x \wedge y) \neq \Pi(x)) \end{cases}$$
  $\Rightarrow \exists x', \exists y'$, *s.t.* $\Pi(x \wedge y) < \Pi(x' \wedge y)$ *and* $\Pi(x \wedge y) < \Pi(x \wedge y')$
  $\Rightarrow \exists x', \exists y'$, *s.t.* $x \wedge y <_\pi x' \wedge y$ *and* $x \wedge y <_\pi x \wedge y'$
  Since the plausibility relation $\geq_\pi$ is Pareto-decomposable, $\exists x', \exists y'$, s.t. $x <_\Pi x'$ and $y <_\Pi y'$ which contradicts proposition 9.

Proof of Proposition 17 *Suppose that this Proposition is false i.e, X and Y are not PO-independent then, we can distinguish two cases:*

- $\exists x, x', \exists y$ *s.t.* $x >_\Pi x'$ *but* $x \wedge y \leq_\pi x' \wedge y$
  $\Rightarrow$ *(i)* $\Pi(x) > \Pi(x')$ *but (ii)* $\Pi(x \wedge y) \leq \Pi(x' \wedge y)$.
  Since M-independence is respected, we have $\Pi(x \wedge y) = \Pi(x)$ and $\Pi(x' \wedge y) = \Pi(x')$. When using these two relations in (ii) we obtain $\Pi(x) \leq \Pi(x')$ which contradicts (i).

- $\exists x, x', \exists y$ *s.t.* $x =_\Pi x'$ *but* $x \wedge y >_\pi x' \wedge y$ *(or* $x \wedge y <_\pi x' \wedge y$*)*
  $\Rightarrow$ *(i)* $\Pi(x) = \Pi(x')$ *but (ii)* $\Pi(x \wedge y) > \Pi(x' \wedge y)$ *(or* $\Pi(x \wedge y) < \Pi(x' \wedge y)$*).*

Since M-independence is respected, we have $\Pi(x \wedge y) = \Pi(x)$ *and* $\Pi(x' \wedge y) = \Pi(x')$.
When using these two relations in (ii) we obtain $\Pi(x) < \Pi(x')$ *which contradicts (i).*

Proof of Proposition 18 *Suppose that this Proposition is false i.e., X and Y are not POS-independent then, the possible situations are:*

- $\exists x, x', \exists y$ *s.t.* $x >_{\Pi} x'$ *but* $x \wedge y \leq_{\pi} x' \wedge y$
  $\Rightarrow$ *(i)* $\Pi(x) > \Pi(x')$ *but (ii)* $\Pi(x \wedge y) \leq \Pi(x' \wedge y)$.

  Since Prod-independence is respected, we have:
  $\Pi(x \wedge y) = \Pi(x) \cdot \Pi(y)$, *and* $\Pi(x' \wedge y) = \Pi(x') \cdot \Pi(y)$.
  When using these two relations in (ii) we obtain:
  $\Pi(x) \cdot \Pi(y) \leq \Pi(x') \cdot \Pi(y)$
  $\Rightarrow \Pi(x) \leq \Pi(x')$ *which contradicts (i).*

- $\exists x, x', \exists y$ *s.t.* $x =_{\Pi} x'$ *but* $x \wedge y >_{\pi} x' \wedge y$ *(or* $x \wedge y <_{\pi} x' \wedge y$*)*
  $\Rightarrow$ *(i)* $\Pi(x) = \Pi(x')$ *but (ii)* $\Pi(x \wedge y) > \Pi(x' \wedge y)$ *(or* $\Pi(x \wedge y) < \Pi(x' \wedge y)$*)*

  Since Prod-independence is respected, we have:
  $\Pi(x \wedge y) = \Pi(x) \cdot \Pi(y)$, *and* $\Pi(x' \wedge y) = \Pi(x') \cdot \Pi(y)$.
  When using these two relations in (ii) we obtain:
  $\Pi(x) \cdot \Pi(y) < \Pi(x') \cdot \Pi(y)$
  $\Rightarrow \Pi(x) < \Pi(x')$ *which contradicts (i).*

Proof of Proposition 20 *Suppose X and Y are NI-independent but not PT-independent. More formally,* $\exists x, \exists y$ *s.t. (i)* $\mathbf{Acc}(x \wedge y) \neq min(\mathbf{Acc}(x), \mathbf{Acc}(y))$. *Hence* $\mathbf{Acc}(x \wedge y) = -1, Acc(x) = 0, Acc(y) = 0$ *(from Proposition 1 item 3). Hence* $\pi(x \wedge y) < 1, \Pi(x) = 1, \Pi(y) = 1$. *This is impossible since NI-independence implies that* $\forall x \in D_X, \forall y \in D_Y, \pi(x \wedge y) = min(\Pi(x), \pi(y))$.

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